

# BOUNDED GEOMETRY AND CHARACTERIZATION OF POST-SINGULARLY FINITE $(p, q)$ -EXPONENTIAL MAPS

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**ABSTRACT.** In this paper we define a topological class of branched covering maps of the plane called *topological exponential maps of type  $(p, q)$*  and denoted by  $\mathcal{TE}_{p,q}$ . We prove that an element  $f \in \mathcal{TE}_{p,q}$  with finite post-singular set is combinatorially equivalent to an entire map of the form  $Pe^Q$ , where  $P$  is a polynomial of degree  $p$  and  $Q$  is a polynomial of degree  $q$  if and only if  $f$  satisfies a *bounded geometry* condition.

## 1. INTRODUCTION

Thurston asked the question “when can we realize a given branched covering map as a holomorphic map in such a way that the post-critical sets correspond?” and answered it for post-critically finite degree  $d$  branched covers of the sphere with  $2 \leq d < \infty$ , [T, DH]. His theorem is that a postcritically finite degree  $d \geq 2$  branched covering of the sphere, with hyperbolic orbifold, is either combinatorially equivalent to a rational map or there is a topological obstruction, now called a “Thurston obstruction”. The rational map is unique up to conjugation by a Möbius transformation.

Thurston’s theorem does not naturally extend to transcendental maps because the proof uses the finiteness of both the degree and the post-critical set in a crucial way. Hubbard, Schleicher, and Shishikura [HSS] generalized Thurston’s theorem to a special infinite degree family they call “exponential type” maps. In that paper, the authors study the limiting behavior of quadratic differentials associated to the exponential functions with finite post-singular set. They use a Levy cycle condition (a special type of Thurston’s topological condition) to characterize when it is possible to realize a given exponential type map with finite post-singular set as an exponential map by combinatorial equivalence.

In this paper, we study a class of entire maps: maps of the form  $Pe^Q$  where  $P$  and  $Q$  are polynomials of degrees  $p$  and  $q$  respectively. This class, which we denote by  $\mathcal{E}_{p,q}$  includes the exponential and polynomials. We call the topological family these analytic maps belong to  $\mathcal{TE}_{p,q}$  and define it below. Thurston’s question makes sense for this family and our main theorem is an answer to this question. Our condition that a post-singularly finite map in  $\mathcal{TE}_{p,q}$  is combinatorially equivalent to an entire map is, however, analytic and not topological. Our approach is to use the “bounded geometry” point of view which we used in our previous study of rational maps in [ZJ,

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CJ1, CJ2] (see [Ji] for an outline and more details) where the bounded geometry condition is an intermediate step that connects various topological obstructions with the characterization of rational maps. The introduction of this intermediate step makes understanding the characterization of rational maps relatively easier and the arguments are more straightforward (see [Ji, CJ1]). The bounded geometry point of view turns out to be useful for characterizing entire and meromorphic maps. We used it previously for the simple family of meromorphic maps with two asymptotic values [CJK].

In this paper we apply our techniques to characterize the larger class of post-singularly finite entire maps with exactly one asymptotic value and finitely many critical points, the model topological space  $\mathcal{T}E_{p,q}$ . Our main result is

**Main Theorem.** *A post-singularly finite map in  $\mathcal{T}E_{p,q}$  is combinatorially equivalent to a post-singularly finite entire map of the form  $Pe^Q$  if and only if it has bounded geometry.*

Our techniques involve adapting the Thurston iteration scheme to our situation. We work with a fixed normalization. There are two important parts to the proof of the main theorem. The first is Theorem 3 which says that the bounded geometry condition implies convergence of the iteration scheme to an entire function of the appropriate type. Its proof is again divided into two parts. The first is an analysis of the topology of pre-images of one of the normalized points and the second is an analysis of the behavior of critical points and critical values for degenerating polynomials. The second part of the proof of the main theorem uses an analysis of quadratic differentials associated to our functions, which together with the compactness, shows that the limit function actually realizes the model map.

The paper is organized as follows. In §2, we review the covering properties of  $(p, q)$ -exponential maps  $E = Pe^Q$ . In §3, we define the family  $\mathcal{T}E_{p,q}$  of  $(p, q)$ -topological exponential maps  $f$ . In §4, we define a topological constraint for the function  $f$  by particular choices of pre-images of the normalized point, and if  $f$  is post-singularly finite, choices of pre-images of the post-singular points. In §5, we define the combinatorial equivalence between post-singularly finite  $(p, q)$ -topological exponential maps. In §6, we define the Teichmüller space  $T_f$  for a post-singularly finite  $(p, q)$ -topological exponential map  $f$  and in §7, we introduce the induced map  $\sigma_f$  from the Teichmüller space  $T_f$  into itself which is the crux of the Thurston iteration scheme. In §8, we define the concept of “bounded geometry” and in §9 we show how bounded geometry implies compactness in  $\mathcal{E}_{p,q}$ . This argument is complicated so in order to clarify the ideas we divide it into two parts. We first prove that bounded geometry implies compactness in  $\mathcal{E}_{0,q}$  for  $1 \leq q < \infty$  and then we give the more general argument to prove bounded geometry implies compactness in  $\mathcal{E}_{p,q}$  for all  $p$  and  $q$ . In §10, we state and prove our main result. In the final section, §11, we make some remarks about the relations between “bounded geometry” and “canonical Thurston obstructions” and between “bounded geometry” and “Levy cycles” in the context of  $\mathcal{T}E_{p,q}$ .

We hope, and are working on eventually showing, that our main theorem can be used as an intermediate step to establishing a topological condition for combinatorial equivalence in this family such as the non-existence of a “Levy cycle” or “canonical Thurston obstruction” (see §11). We note that even for finite degree covering maps the “Thurston obstruction” or “Levy cycle” condition has the serious limitation that the criteria are not easy to check, even though they are purely combinatorial-topological.

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## 2. THE SPACE $\mathcal{E}_{p,q}$ OF $(p, q)$ -EXPONENTIAL MAPS

We use the following notation:  $\mathbb{C}$  is the complex plane,  $\hat{\mathbb{C}}$  is the Riemann sphere and  $\mathbb{C}^*$  is the complex plane punctured at the origin.

A  $(p, q)$ -*exponential map* is an entire function of the form  $E = Pe^Q$  where  $P$  and  $Q$  are polynomials of degrees  $p \geq 0$  and  $q \geq 0$  respectively such that  $p + q \geq 1$ . We use the notation  $\mathcal{E}_{p,q}$  for the set of  $(p, q)$ -exponential maps.

Note that if  $P(z) = a_0 + a_1z + \dots a_pz^p$ ,  $Q(z) = b_0 + b_1z + \dots b_qz^q$ ,  $\hat{P}(z) = e^{b_0}P(z)$  and  $\hat{Q}(z) = Q(z) - b_0$  then

$$P(z)e^{Q(z)} = \hat{P}e^{\hat{Q}(z)}.$$

To avoid this ambiguity we always assume  $b_0 = 0$ . If  $q = 0$ , then  $E$  is a polynomial of degree  $p$ . Otherwise,  $E$  is a transcendental entire function with essential singularity at infinity.

The growth rate of an entire function  $f$  is defined as

$$\limsup_{r \rightarrow \infty} \frac{\log \log M(r)}{\log r}$$

where  $M(r) = \sup_{|z|=r} |f(z)|$ . It is easy to see that the growth rate of  $E$  is  $q$ .

Recall that an asymptotic tract  $V$  for an entire transcendental function  $g$  is a simply connected unbounded domain such that  $g(V) \subset \hat{\mathbb{C}}$  is conformally a punctured disk  $D \setminus \{a\}$  and the map  $g : V \rightarrow g(V)$  is a universal cover. The point  $a$  is the asymptotic value corresponding to the tract. For functions  $E$  in  $\mathcal{E}_{p,q}$  we have

**Proposition 1.** *If  $q \geq 1$ ,  $E$  has  $2q$  distinct asymptotic tracts that are separated by  $2q$  rays. Each tract maps to a punctured neighborhood of either zero or infinity and these are the only asymptotic values.*

*Proof.* From the growth rate of  $E$  we see that for  $|z|$  large, the behavior of the exponential dominates. Since  $Q(z) = b_q z^q + \text{lower order terms}$ , in a neighborhood of infinity there are  $2q$  branches of  $\Re Q = 0$  asymptotic to equally spaced rays. In the  $2q$  sectors defined by these rays the signs of  $\Re Q$  alternate. If  $\gamma(t)$  is a curve such that  $\lim_{t \rightarrow \infty} \gamma(t) = \infty$  and  $\gamma(t)$  stays in one sector for all large  $t$ , then either  $\lim_{t \rightarrow \infty} E(\gamma(t)) = 0$  or  $\lim_{t \rightarrow \infty} E(\gamma(t)) = \infty$ , as  $\Re Q$  is negative or positive in the sector. It follows that there are exactly  $q$  sectors that are asymptotic tracts for 0 and  $q$  sectors that are asymptotic tracts for infinity. Because the complement of these tracts in a punctured neighborhood of infinity consists entirely of these rays, there can be no other asymptotic tracts.  $\square$

**Remark 1.** *The directions dividing the asymptotic tracts are called Julia rays or Julia directions for  $E$ . If  $\gamma(t)$  tends to infinity along a Julia ray,  $E(\gamma(t))$  remains in a compact domain in the plane. It spirals infinitely often around the origin.*

Two  $(p, q)$ -exponential maps  $E_1$  and  $E_2$  are conformally equivalent if they are conjugate by a conformal automorphism  $M$  of the Riemann sphere  $\hat{\mathbb{C}}$ , that is,  $E_1 = M \circ E_2 \circ M^{-1}$ . The automorphism  $M$  must be a Möbius transformation and it must fix both 0 and  $\infty$  so that it must be the affine stretch map  $M(z) = az$ ,  $a \neq 0$ . We are interested in conformal equivalence classes of maps, so by abuse of notation, we treat conformally equivalent  $(p, q)$ -exponential maps  $E_1$  and  $E_2$  as the same.

The critical points of  $E = Pe^Q$  are the roots of  $P' + PQ' = 0$ . Therefore,  $E$  has  $p + q - 1$  critical points counted with multiplicity which we denote by

$$\Omega_E = \{c_1, \dots, c_{p+q-1}\}.$$

Note that if  $E = 0$  then  $P = 0$ . This in turn implies that if  $c \in \Omega_E$  maps to 0, then  $c$  must also be a critical point of  $P$ . Since  $P$  has only  $p - 1$  critical points counted with multiplicity, there must be at least  $q$  points (counted with multiplicity) in  $\Omega$  which are not mapped to 0. Denote by

$$\Omega_{E,0} = \{c_1, \dots, c_k\}, \quad k \leq p - 1,$$

the (possibly empty) subset of  $\Omega_E$  consisting of critical points such that  $E(c_i) = 0$ . Denote its complement in  $\Omega_E$  by

$$\Omega_{E,1} = \Omega_E \setminus \Omega_{E,0} = \{c_{k+1}, \dots, c_{p+q-1}\}.$$

When  $q = 0$ ,  $E$  is a polynomial. The *post-singular set* in this special case is the same as the *post-critical set*. It is defined as

$$P_E = \overline{\cup_{n \geq 1} E^n(\Omega_E)} \cup \{\infty\}.$$

To avoid trivial cases here we will assume that  $\#(P_E) \geq 4$ . Conjugating by an affine map  $z \rightarrow az + b$  of the complex plane, we normalize so that  $0, 1 \in P_E$ .

When  $q = 1$  and  $p = 0$ ,  $\Omega_E = \emptyset$  and  $\mathcal{E}_{0,1}$  consists of exponential maps  $\lambda e^z$ ,  $\lambda \in \mathbb{C}^*$ . The *post-singular set* in this special case is defined as

$$P_E = \overline{\cup_{n \geq 0} E^n(0)} \cup \{\infty\}.$$

Conjugating by an affine stretch  $z \mapsto az$  of the complex plane, we normalize so that  $E(0) = 1$ . Note that the family  $\lambda e^z$  after this normalization takes the form  $e^{\lambda z}$ .

When  $q \geq 2$  and  $p = 0$  or when  $q \geq 1$  and  $p \geq 1$ ,  $\Omega_{E,1}$  is a non-empty set. Let

$$\mathcal{V} = E(\Omega_{E,1}) = \{v_1, \dots, v_m\}$$

denote the set of non-zero critical values of  $E$ . The *post-singular set* for  $E$  in the general case is now defined as

$$P_E = \overline{\cup_{n \geq 0} E^n(\mathcal{V} \cup \{0\})} \cup \{\infty\}.$$

We normalize as follows:

If  $E$  does not fix 0, which is always true if  $q \geq 2$  and  $p = 0$ , we conjugate by an affine stretch  $z \rightarrow az$  so that  $E(0) = 1$ .

If  $E(0) = 0$ , there is a critical point in  $c_{k+1}$  in  $\Omega_{E,1}$  with  $c_{k+1} \neq 0$  and  $v_1 = E(c_{k+1}) \neq 0$ . In this case we normalize so that  $v_1 = 1$ . The family  $\mathcal{E}_{1,1}$  consists of functions of the form  $\lambda z e^{\mu z}$ . After normalization they take the form

$$\lambda z e^{-\lambda z/e}.$$

### 3. TOPOLOGICAL EXPONENTIAL MAPS OF TYPE $(p, q)$

We use the notation  $\mathbb{R}^2$  for the Euclidean plane. We define the space  $\mathcal{T}E_{p,q}$  of *topological exponential maps of type  $(p, q)$*  with  $p + q \geq 1$ . These are branched coverings with a single finite asymptotic value, normalized to be at zero, modeled on the maps in the holomorphic family  $\mathcal{E}_{p,q}$ . For a full discussion of the covering properties for this family see Zakeri [Z], and for a more general discussion of maps with finitely many asymptotic and critical values see Nevanlinna [N].

If  $q = 0$ , then  $\mathcal{T}E_{p,0}$  consists of all topological polynomials  $P$  of degree  $p$ : these are degree  $p$  branched coverings of the sphere such that  $f^{-1}(\infty) = \{\infty\}$ .

If  $q = 1$  and  $p = 0$ , the space  $\mathcal{T}E_{0,1}$  consists of universal covering maps  $f : \mathbb{R}^2 \rightarrow \mathbb{R}^2 \setminus \{0\}$ . These are discussed at length in [HSS] where they are called topological exponential maps.

The polynomials  $P$  and  $Q$  contribute differently to the covering properties of maps in  $\mathcal{E}_{p,q}$ . As we saw, the degree of  $Q$  controls the growth and behavior at infinity. Using maps  $e^Q$  as our model we first define the space  $\mathcal{T}E_{0,q}$ .

**Definition 1.** *If  $q \geq 2$  and  $p = 0$ , the space  $\mathcal{T}E_{0,q}$  consists of topological branched covering maps  $f : \mathbb{R}^2 \rightarrow \mathbb{R}^2 \setminus \{0\}$  satisfying the following conditions:*

- i) *The set of branch points,  $\Omega_f = \{c \in \mathbb{R}^2 \mid \deg_c f \geq 2\}$  consists of  $q - 1$  points counted with multiplicity.*
- ii) *Let  $\mathcal{V} = \{v_1, \dots, v_m\} = f(\Omega_f) \subset \mathbb{R}^2 \setminus \{0\}$  be the set of distinct images of the branch points. For  $i = 1, \dots, m$ , let  $L_i$  be a smooth topological ray in  $\mathbb{R}^2 \setminus \{0\}$  starting at  $v_i$  and extending to  $\infty$  such that the collection of rays  $\{L_1, \dots, L_m\}$  are pairwise disjoint. Then*

- (1)  $f^{-1}(L_i)$  consists of infinitely many rays starting at points in the preimage set  $f^{-1}(v_i)$ . If  $x \in f^{-1}(v_i) \cap \Omega_f$ , there are  $d_x = \deg_x f$  rays meeting at  $x$  called critical rays. If  $x \in f^{-1}(v_i) \setminus \Omega_f$ , there is only one ray emanating from  $x$ ; it is called a non-critical ray. Set

$$W = \mathbb{R}^2 \setminus (\cup_{i=1}^m L_i \cup \{0\}).$$

- (2) The set of critical rays meeting at points in  $\Omega_f$  divides  $f^{-1}(W)$  into  $q = 1 + \sum_{c \in \Omega_f} (d_c - 1)$  open unbounded connected components  $W_1, \dots, W_q$ .
- (3)  $f : W_i \rightarrow W$  is a universal covering for each  $1 \leq i \leq q$ .

Note that the map restricted to each  $W_i$  is a topological model for the exponential map  $z \mapsto e^z$  and the local degree at the critical points determines the number of  $W_i$  attached at the point.

We now define the space  $\mathcal{T}E_{p,q}$  in full generality where we assume  $p > 0$  and there is additional behavior modeled on the role of the new critical points of  $Pe^Q$  introduced by the non-constant polynomial  $P$ .

**Definition 2.** If  $q \geq 1$  and  $p \geq 1$ , the space  $\mathcal{T}E_{p,q}$  consists of topological branched covering maps  $f : \mathbb{R}^2 \rightarrow \mathbb{R}^2$  satisfying the following conditions:

- i)  $f^{-1}(0)$  consists of  $p$  points counted with multiplicity.
- ii) The set of branch points,  $\Omega_f = \{c \in \mathbb{R}^2 \mid \deg_c f \geq 2\}$  consists of  $p + q - 1$  points counted with multiplicity.
- iii) Let  $\Omega_{f,0} = \Omega_f \cap f^{-1}(0)$  be the  $k < p$  branch points that map to 0 and  $\Omega_{f,1} = \Omega_f \setminus \Omega_{f,0}$  the  $p + q - 1 - k$  branch points that do not. Note that  $\Omega_{f,1}$  contains at least  $q$  points and  $\mathcal{V} = \{v_1, \dots, v_m\} = f(\Omega_{f,1})$  is contained in  $\mathbb{R}^2 \setminus \{0\}$ . For  $i = 1, \dots, m$ , let  $L_i$  be a smooth topological ray in  $\mathbb{R}^2 \setminus \{0\}$  starting at  $v_i$  and extending to  $\infty$  such that the collection of rays  $\{L_1, \dots, L_m\}$  are pairwise disjoint. Then

- (1)  $f^{-1}(L_i)$  consists of infinitely many rays starting at points in the preimage set  $f^{-1}(v_i)$ . If  $x \in f^{-1}(v_i) \setminus \Omega_{f,1}$ , there is only one ray emanating from  $x$ ; this is a non-critical ray. If  $x \in f^{-1}(v_i) \cap \Omega_{f,1}$ , there are  $d_x = \deg_x f$  critical rays meeting at  $x$ . Set

$$W = \mathbb{R}^2 \setminus (\cup_{i=1}^m (L_i) \cup \{0\}).$$

- (2) The collection of all critical rays meeting at points in  $\Omega_{f,1}$  divides  $f^{-1}(W)$  into  $l = p + q - k = 1 + \sum_{c \in \Omega_{f,1}} (d_c - 1)$  open unbounded connected components.
- (3) Set  $f^{-1}(0) = \{a_i\}_{i=1}^{p-k}$  where the  $a_i$  are distinct. Each  $a_i$  is contained in a distinct component of  $f^{-1}(W)$ ; label these components  $W_{i,0}$ ,  $i = 1, \dots, p - k$ . Then the restriction  $f : W_{i,0} \setminus \{a_i\} \rightarrow W$  is an unbranched covering map of degree  $d_i = \deg_{a_i} f$  where  $d_i > 1$  if  $a_i \in \Omega_{f,0}$  and  $d_i = 1$  otherwise.
- (4) Label the remaining  $q$  connected components of  $f^{-1}(W)$  by  $W_{j,1}$ ,  $j = 1, \dots, q$ . Then the restriction  $f : W_{j,1} \rightarrow W$  is a universal covering map.

In Figure 1 we have drawn the configuration for  $f(z) = (z+1)e^{(1+4\pi i)z^2}$ . The larger points are the critical points and the smaller points are zero, 1 and the critical values. For computational reasons the curves are the pre-images of full lines through the critical values and zero rather than rays.

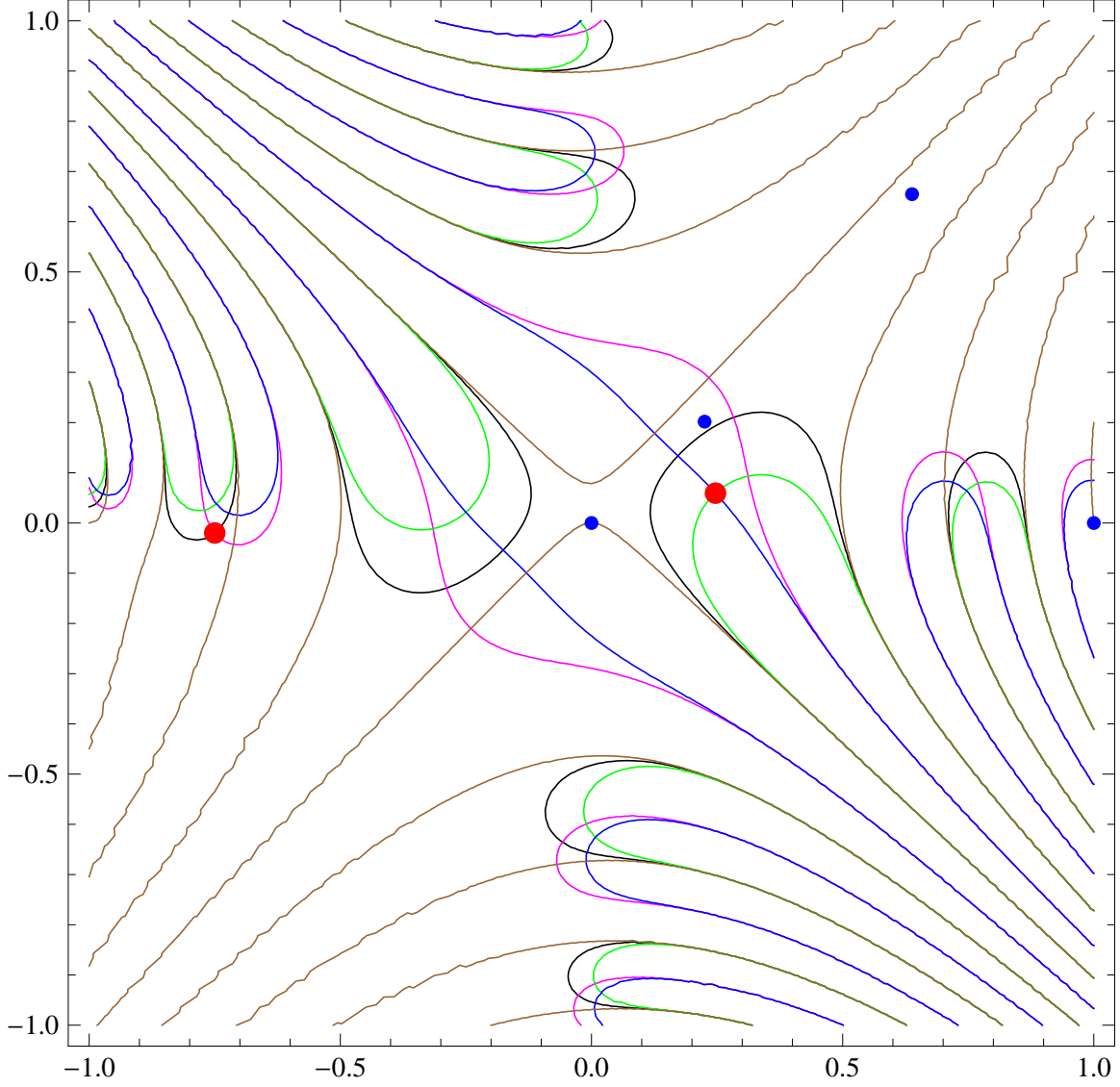


FIGURE 1. The lines  $f^{-1}(L_i)$  for  $f(z) = (z+1)e^{(1+4\pi i)z^2}$  with critical points and singular values

In section 3 of [Z], Zakeri proves that the  $(p, q)$ -exponential maps are topological exponential maps of type  $(p, q)$ . The converse is also true.

**Theorem 1.** *Suppose  $f \in \mathcal{TE}_{p,q}$  is analytic. Then  $f = Pe^Q$  for two polynomials  $P$  and  $Q$  of degrees  $p$  and  $q$ . That is, an analytic topological exponential map of type  $(p, q)$  is a  $(p, q)$ -exponential map.*

*Proof.* If  $q = 0$ , then  $f$  is a polynomial  $P$  of degree  $p$ .

If  $q \geq 1$ , then  $f$  is an entire function with  $p$  roots, counted with multiplicity. Every such function can be expressed as

$$f(z) = P(z)e^{g(z)}$$

where  $P$  is a polynomial of degree  $p$  and  $g$  is some entire function (see [Al, Section 2.3]).

Consider

$$f'(z) = (P(z)g'(z) + P'(z))e^{g(z)}.$$

It is also an entire function, and by assumption it has  $p+q-1$  roots so that  $Pg' + P'$  is a polynomial of degree  $p+q-1$ . It follows that  $g'$  is a polynomial of degree  $q-1$  and  $g = Q$  is a polynomial of degree  $q$ .  $\square$

Note that if  $f \in \mathcal{TE}_{p,q}$ ,  $q \neq 0$ , the origin plays a special role: it is the only point with finitely many pre-images. The conjugate of  $f$  by  $z \mapsto az$ ,  $a \in \mathbb{C}^*$ , is also in  $\mathcal{TE}_{p,q}$ ; we call conjugate maps equivalent

For  $f \in \mathcal{TE}_{p,q}$ , we define the *post-singular set* as follows:

- i) When  $q = 0$ ,  $E$  is a polynomial and, as mentioned in the introduction, is treated elsewhere. We therefore always assume  $q \geq 1$ .
- ii) When  $q = 1$  and  $p = 0$ , the *post-singular set* is

$$P_f = \overline{\cup_{n \geq 0} f^n(0)} \cup \{\infty\}.$$

We normalize so that  $f(0) = 1 \in P_f$ .

- iii) When  $q \geq 1$  and  $p \geq 1$ , the set of branch points is

$$\Omega_f = \{c \in \mathbb{R}^2 \mid \deg_c f \geq 2\}$$

and the *post-singular set* is

$$P_f = \overline{\cup_{n \geq 0} f^n(\mathcal{V} \cup \{0\})} \cup \{\infty\}.$$

If  $q > 1$  or if  $q = 1$  and  $f(0) \neq 0$ , we normalize so that  $f(0) = 1 \in P_f$ . If  $f(0) = 0$ , then, by assumption, there is a branch point  $c_{k+1} \neq 0$  such that  $v_1 = f(c_{k+1}) \neq 0$ . We normalize so that  $v_1 = 1$ .

To avoid trivial cases we assume that  $\#(P_f) \geq 4$ .

It is clear that, in any case,  $P_f$  is forward invariant, that is,

$$f(P_f \setminus \{\infty\}) \cup \{\infty\} \subseteq P_f$$

or equivalently,

$$f^{-1}(P_f \setminus \{\infty\}) \cup \{\infty\} \supset P_f.$$

Note that since we assume  $q \geq 1$ ,  $f^{-1}(P_f \setminus \{\infty\}) \setminus (P_f \setminus \{\infty\})$  contains infinitely many points.



**Definition 3.** We call  $f \in \mathcal{TE}_{p,q}$  post-singularly finite if  $\#(P_f) < \infty$ .

#### 4. TOPOLOGICAL CONSTRAINTS

Consider  $f \in \mathcal{TE}_{p,q}$ . By our normalization, the asymptotic value is 0 and either  $f(0) = 1$  or  $f(0) = 0$  and  $f(c) = 1$  for some critical point  $c$ . In either case, the point 1 belongs to  $P_f$ . We will define a topological constraint for  $f$  based on a path from 0 to  $f(1)$  that passes through 1. Up to homotopy relative to  $P_f$ , there are many choices for such a path but once a path is picked, it will be fixed in the rest of the paper. This choice gives us a partial marking of the space  $(\mathbb{R}^2 \cup \{\infty\}) \setminus P_f$ .

Recall the notation  $\{L_i\}$ ,  $W$ , and  $\{W_{i,0}, W_{1,j}\}$  from section 3. We saw there that  $f : W_{i,0} \rightarrow W$  is a branched covering of finite degree with only one branch point  $a_{i,0} \in \Omega_{f,0}$ . Each  $W_{i,0}$  is bounded by curves in  $f^{-1}(L_i)$  and is the union of finitely many fundamental domains for  $f$ . Each  $W_{j,1}$  is a universal covering of  $W$ , is bounded by curves in  $f^{-1}(L_i)$  and is the union of infinitely many fundamental domains for  $f$ . The Julia direction is the asymptotic direction of a path extending to  $\infty$  that crosses all these fundamental domains transversally. Note that the fundamental domains for each  $W_{j,1}$  depend on a choice of a path from the asymptotic value to infinity.

First assume  $f(0) = 1$ . Let  $\gamma_0$  be a smooth curve connecting 0 and 1 in  $\mathbb{R}^2$  disjoint from  $\Omega_f \cup P_f$ , except for its endpoints 0 and 1, and such that its intersection with any curve in  $f^{-1}(L_i)$  is a single point. Assume further that  $\gamma_0$  traverses at least one full fundamental domain for  $f$  in the component of  $f^{-1}(W)$  containing 0; it can only pass through finitely many fundamental domains since its image is a compact subset in  $\mathbb{R}^2$ . The image curve  $f(\gamma_0)$  is a continuous curve that spirals around 0 but, is in general, not closed. Adjusting the choice of fundamental domains if necessary, the point 0 lies inside some  $W_{i,0}$  or  $W_{j,1}$  and inside some fundamental domain  $F$  of  $W_{i,0}$  or  $W_{j,1}$ . There is a unique point of  $f^{-1}(f(1))$  in  $F$ , which we denote by  $a$ . Let  $\gamma_1$  be a smooth curve wholly contained in  $F$  joining 0 to  $a$  and set  $\gamma = \gamma_0 \cup \gamma_1$ . The image  $f(\gamma)$  has both endpoints at  $f(1) = f(a)$  and so is a closed curve.

In the following figures we illustrate this with the function  $E(z) = (z+1)e^{(1+4\pi i)z^2}$ . Note that  $E(1) = 2e$ . In Figure 2 we see pre-images of the real axis and the vertical line,  $\Re z = 2e$ . The points marked are zero, 1 and the point  $a = f^{-1}(f(1))$  in a fundamental domain containing zero. We take  $\gamma_0$  as the line joining 1 to 0 and  $\gamma_1$  as the line joining 0 to  $a$ . In Figure 3 we have drawn the closed curve  $f(\gamma)$ . We have marked the points 1 and  $f(1)$ . We see that it spirals around the origin with winding number 2. This is a reflection of the  $4\pi i$  in the coefficient of  $Q$ .

If  $f(0) = 0$ ,  $f$  is normalized so that there is a critical point  $c \neq 0$  such that  $f(c) = 1$  and there is a similar construction for the curve  $\gamma$  with  $c$  playing the role of 0. Now let  $\gamma_0$  be a smooth curve connecting  $c$  and 1 in  $\mathbb{R}^2$  disjoint from  $\Omega_f \cup P_f$ , except for its endpoints  $c$  and 1, and such that its intersection with any curve in  $f^{-1}(L_i)$  is a single point. The construction proceeds exactly as above. Assume that  $\gamma_0$  traverses at least one fundamental domain fully and, again, it can only pass through finitely many fundamental domains. The curve  $f(\gamma_0)$  is a continuous curve

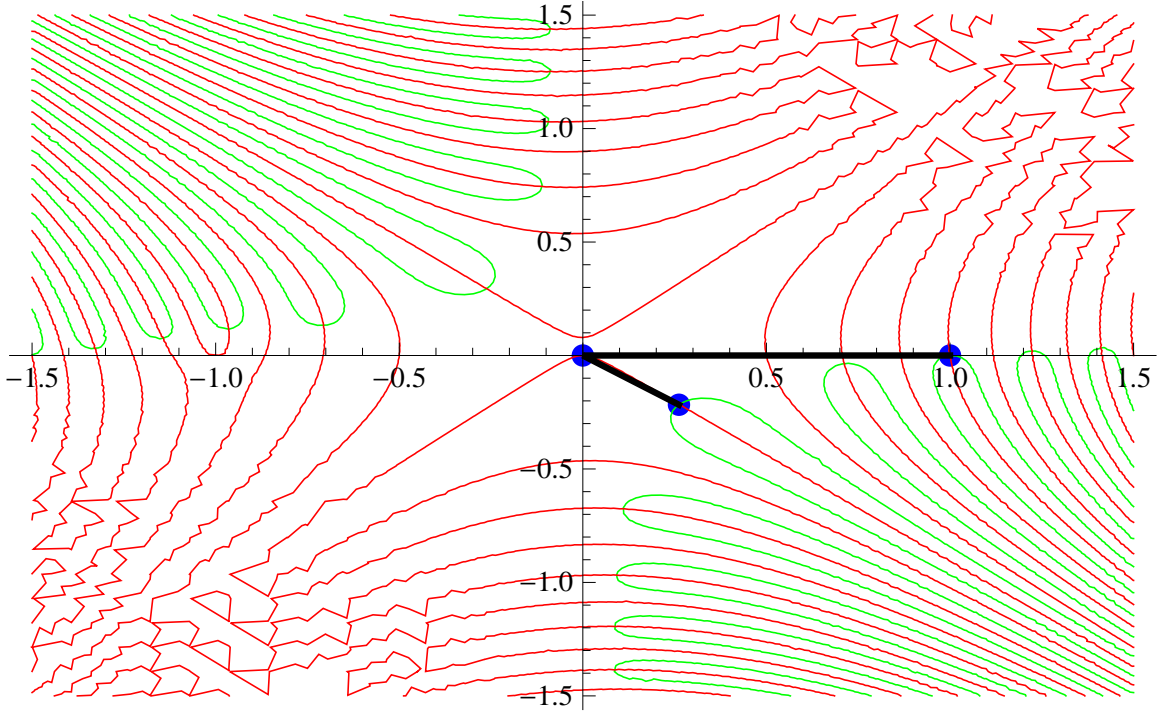


FIGURE 2. The curve  $\gamma$  for  $E^{-1}(E(1))$  for  $E(z) = (z + 1)e^{(1+4\pi i)z^2}$ .

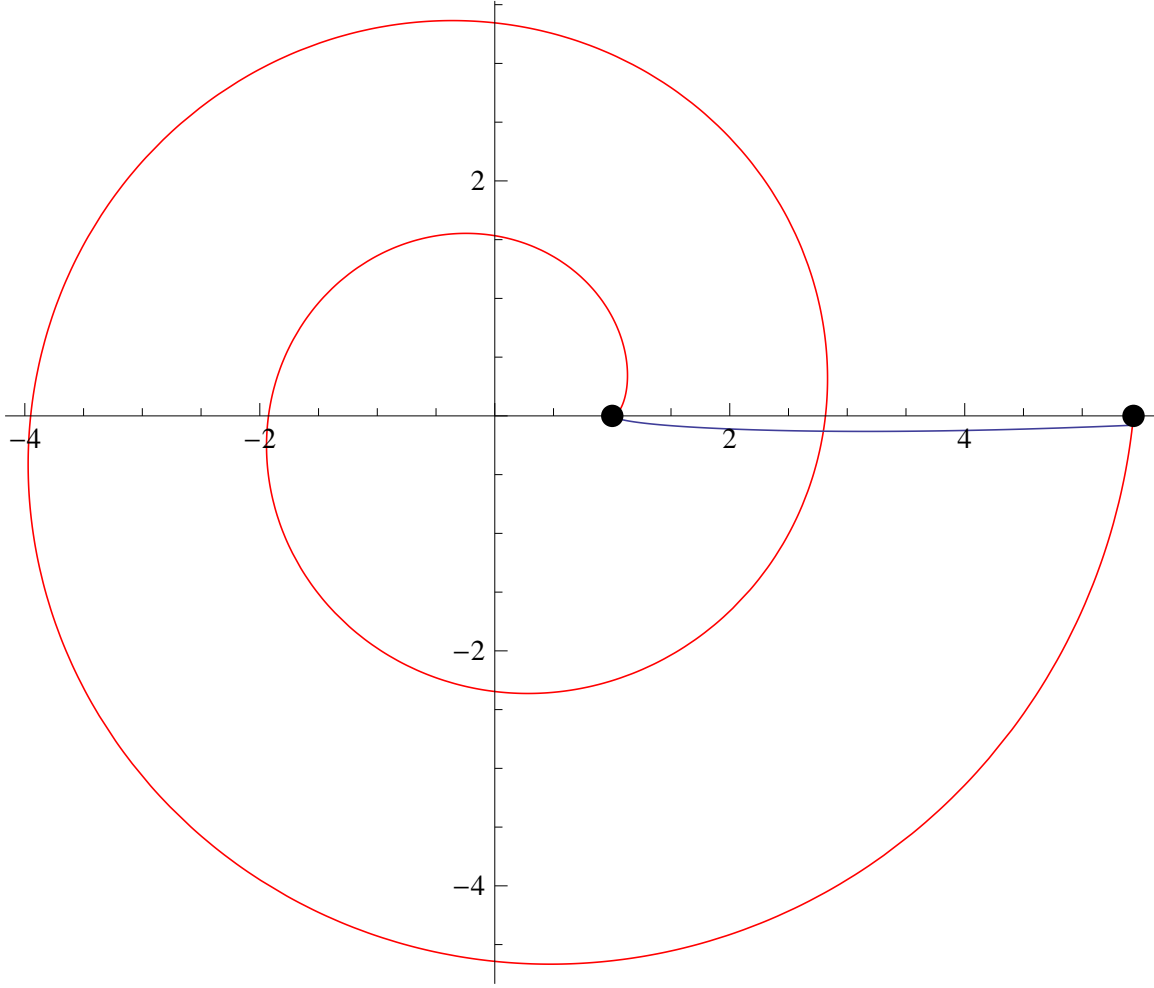
spiraling around 0 but, in general, is not closed. Consider the fundamental domain  $F$  containing  $c$ . In  $F$  there is a unique point of  $f^{-1}(1)$ , which we denote by  $a$ . Again let  $\gamma_1$  be a smooth curve wholly contained in  $F$  joining  $c$  to  $a$  and set  $\gamma = \gamma_0 \cup \gamma_1$ . The image  $f(\gamma)$  has both endpoints at  $f(1) = f(a)$  and so is a closed curve.

In either case, the winding number  $\eta_0$  of the curve  $f(\gamma)$  is a topological invariant. Thus, if  $E \in \mathcal{E}_{p,q}$  is topologically conjugate to  $f$ , the winding number puts a constraint on the coefficients of  $Q$ ; in particular, they are bounded in the Julia directions.

In section 7 we define the Thurston map and in section 9, given  $f$ , we iterate the map to generate a sequence of functions  $E_n = P_n e^{Q_n}$ . In this iteration process the  $E_n$  are all topologically conjugate to  $f$  so the winding number is the same for all of them and coefficients of the  $Q_n$  are restricted. This constraint plays an important role in the proof of our main theorem.

If  $f \in \mathcal{T}_{E_{p,q}}$  is post-singularly finite there are additional topological invariants. The asymptotic value 0 and the critical values in  $\mathcal{V}$  are periodic or pre-periodic so that  $P_f$  contains  $r$  distinct periodic cycles and each cycle has period  $n_i$ ,  $i = 1, \dots, r$ . Denote the cycles by

$$\{x_{i,1}, x_{i,2} = f(x_{i,1}), \dots, x_{i,n_i+1} = f(x_{i,n_i}) = x_{i,1}\}, \quad i = 1, \dots, r,$$

FIGURE 3. The curve  $f(\gamma)$  for  $E(z) = (z + 1)e^{(1+4\pi i)z^2}$ 

where each  $x_{i,1} = f^{k_i}(v)$  for some  $v \in \mathcal{V}$  and  $k_i > 0$ , and  $x_{i,0} = f^{k_i-1}(v)$  does not belong to the cycle.

For each  $i$ ,  $x_{i,0}$  belongs to the closure of one of the components  $W_{i,0}$  of  $f^{-1}(W)$ . Moreover, it is a particular pre-image in this component; we need to keep track of this information. We do this by defining closed curves  $\gamma_i$  in essentially the same way we did above. The choices we make in the construction are tantamount to a marking of the space  $(\mathbb{R}^2 \cup \{\infty\}) \setminus P_f$ .

For each of the pairs of points  $(x_{i,0}, x_{i,1})$ , we choose a smooth curve  $\gamma_{i,0}$  in  $\mathbb{R}^2$  joining them so that  $\gamma_{i,0}$  is disjoint from  $\Omega_f \cup P_f$  except at its endpoints  $x_{i,0}$  and  $x_{i,1}$  and such that it intersects any curve in  $\{f^{-1}(L_i)\}$  in at most one point. As above, we assume that  $\gamma_{i,0}$  traverses at least one fundamental domain in  $W_{i,0}$  fully and note that since its image is a compact subset in  $\mathbb{R}^2$ , it can only pass through finitely many

fundamental domains. The image  $f(\gamma_{i,0})$  is a continuous curve spiraling around 0 but is not, in general, a closed curve. Consider the fundamental domain  $F_i$  containing  $x_{i,0}$ . In  $F_i$  there is a unique point of  $f^{-1}(x_{i,1})$ , which we denote by  $a_i$ . We extend  $\gamma_{i,0}$  to a smooth curve  $\gamma_i$  with endpoints  $x_{i,1}$  and  $a_i$  by joining  $x_{i,0}$  and  $a_i$  by a curve wholly contained in  $F_i$  so that the curve  $f(\gamma_i)$  is a closed curve. This closed curve has a well-defined winding number  $\eta_i$  about the origin. The winding numbers  $\eta_i$  are additional topological invariants; they depend only on the choice of the curves  $\gamma_i$ . The full set of winding numbers,  $\eta_f = \{\eta_0, \eta_1, \dots, \eta_r\}$  is an invariant of the topological conjugacy class of  $f$ . This invariant controls the coefficients of  $Q$  for  $E = Pe^Q \in E_{p,q}$  conjugate to  $f$ ; as we noted above, this constraint will play an important role in the Thurston iteration process later in the paper.

We note that in the special case where  $f$  is an exponential type map, that is  $f \in \mathcal{TE}_{0,1}$ , there is only one periodic cycle and the winding number counts the number of periodic strips (or fundamental domains) between  $x_0$  and  $x_{n-1}$ . Our marking in this case is equivalent to the one given in [HSS] even though there the periods are all fixed whereas in our normalization the periods depend on the parameters.

## 5. COMBINATORIAL EQUIVALENCE

**Definition 4.** Suppose  $f, g$  are two post-singularly finite maps in  $\mathcal{TE}_{p,q}$ . We say that they are combinatorially equivalent if there are homeomorphisms  $\phi$  and  $\psi$  of the sphere  $S^2 = \mathbb{R}^2 \cup \{\infty\}$  fixing 0 and  $\infty$  such that  $\phi \circ f = g \circ \psi$  on  $\mathbb{R}^2$  and  $\phi^{-1} \circ \psi$  is isotopic to the identity of  $S^2$  rel  $P_f$ .

The commutative diagram for the above definition is

$$\begin{array}{ccc} \mathbb{R}^2 & \xrightarrow{\psi} & \mathbb{R}^2 \\ \downarrow f & & \downarrow g \\ \mathbb{R}^2 & \xrightarrow{\phi} & \mathbb{R}^2 \end{array}$$

Consider  $\mathbb{R}^2 \cup \{\infty\}$  equipped with the standard conformal structure as the Riemann sphere. Then  $f \in \mathcal{TE}_{p,q}$  is a map from  $\hat{\mathbb{C}}$  into itself. We say  $f \in \mathcal{TE}_{p,q}$  is *locally  $K$ -quasiconformal* for some  $K > 1$  if for any  $z \in \hat{\mathbb{C}} \setminus \Omega_f \cup \{0\}$ , there is a neighborhood  $U$  of  $z$  such that  $f : U \rightarrow f(U)$  is  $K$ -quasiconformal.

**Lemma 1.** Any post-singularly finite  $f \in \mathcal{TE}_{p,q}$  is combinatorially equivalent to some locally  $K$ -quasiconformal map  $g \in \mathcal{TE}_{p,q}$ .

*Proof.* Let  $D_0$  be a disk in  $\mathbb{R}^2$  so large that it contains  $\Omega_f \cup P_f \setminus \{\infty\}$ . Since  $\bar{D}_0$  is compact,  $f$  restricted to  $D_0$  has finite degree. Therefore we can find a map  $\tilde{f}_0 : \bar{D}_0 \rightarrow f(D_0)$  such that  $\tilde{f}_0(z) = f(z)$  for  $z \in \Omega_f \cup P_f \setminus \{\infty\} \cup \partial D$  and  $\tilde{f}_0$  is isotopic to  $f$  and locally  $K_0$ -quasiconformal for some  $K_0$  for all other  $z \in D_0$ . The complement of  $D_0$  in  $\mathbb{R}$  consists of domains  $D_i$ ,  $i = 1, \dots, q$  each a universal cover of a punctured neighborhood of zero, separated by  $q$  curves tending to infinity. On

each  $D_i$ , we can find a  $K_i$ -quasiconformal map  $\phi_i$  such that  $\tilde{f}_i = \exp \circ \phi$  is isotopic to  $f$  and  $\tilde{f} = f$  on  $\partial D \cap V_i$ . Set  $g = \tilde{f}_i$  on each  $D_i$ ,  $i = 0, \dots, q$ ; then  $g$  is isotopic to  $f$  on  $\mathbb{R}^2 \text{ rel } P_f$  and  $g$  is  $K$ -quasiconformal for  $K = \max K_i$ .  $\square$

Thus without loss of generality, in the rest of the paper, we will assume that any post-singularly finite  $f \in \mathcal{TE}_{p,q}$  is locally  $K$ -quasiconformal for some  $K > 1$ .

## 6. TEICHMÜLLER SPACE $T_f$

Recall that we denote  $\mathbb{R}^2 \cup \{\infty\}$  equipped with the standard conformal structure by  $\hat{\mathbb{C}}$ . Let  $\mathcal{M} = \{\mu \in L^\infty(\hat{\mathbb{C}}) \mid \|\mu\|_\infty < 1\}$  be the unit ball in the space of all measurable functions on the Riemann sphere. Each element  $\mu \in \mathcal{M}$  is called a Beltrami coefficient. For each Beltrami coefficient  $\mu$ , the Beltrami equation

$$w_{\bar{z}} = \mu w_z$$

has a unique quasiconformal solution  $w^\mu$  which maps  $\hat{\mathbb{C}}$  to itself fixing  $0, 1, \infty$ . Moreover,  $w^\mu$  depends holomorphically on  $\mu$ .

Let  $f$  be a post-singularly finite map in  $\mathcal{TE}_{p,q}$  with post-singular set  $P_f$ . The Teichmüller space  $T(\hat{\mathbb{C}}, P_f)$  is defined as follows. Given Beltrami differentials  $\mu, \nu \in \mathcal{M}$  we say that  $\mu$  and  $\nu$  are equivalent, and denote this by  $\mu \sim \nu$ , if  $(w^\nu)^{-1} \circ w^\mu$  is isotopic to the identity map of  $\hat{\mathbb{C}} \text{ rel } P_f$ . The equivalent class of  $\mu$  under  $\sim$  is denoted by  $[\mu]$ . We set

$$T_f = T(\hat{\mathbb{C}}, P_f) = \mathcal{M} / \sim.$$

It is easy to see that  $T_f$  is a finite-dimensional complex manifold and is equivalent to the classical Teichmüller space  $\text{Teich}(\hat{\mathbb{C}} \setminus P_f)$  of Riemann surfaces with basepoint  $\hat{\mathbb{C}} \setminus P_f$ . Therefore, the Teichmüller distance  $d_T$  and the Kobayashi distance  $d_K$  on  $T_f$  coincide.

## 7. INDUCED HOLOMORPHIC MAP $\sigma_f$

For any post-singularly finite  $f$  in  $\mathcal{TE}_{p,q}$ , there is an induced map  $\sigma = \sigma_f$  from  $T_f$  into itself given by

$$\sigma([\mu]) = [f^* \mu],$$

where

$$(1) \quad f^* \mu(z) = \frac{\mu_f(z) + \mu_f((f(z))\theta(z))}{1 + \overline{\mu_f(z)} \mu_f(f(z))\theta(z)}, \quad \theta(z) = \frac{\bar{f}_z}{f_z}.$$

It is a holomorphic map so that

**Lemma 2.** *For any two points  $\tau$  and  $\tilde{\tau}$  in  $T_f$ ,*

$$d_T(\sigma(\tau), \sigma(\tilde{\tau})) \leq d_T(\tau, \tilde{\tau}).$$

The next lemma follows directly from the definitions.

**Lemma 3.** *A post-singularly finite  $f$  in  $\mathcal{TE}_{p,q}$  is combinatorially equivalent to a  $(p, q)$ -exponential map  $E = Pe^Q$  iff  $\sigma$  has a fixed point in  $T_f$ .*

## 8. BOUNDED GEOMETRY

For any  $\tau_0 \in T_f$ , let  $\tau_n = \sigma^n(\tau_0)$ ,  $n \geq 1$ . The iteration sequence  $\tau_n = [\mu_n]$  determines a sequence of finite subsets

$$P_{f,n} = w^{\mu_n}(P_f), \quad n = 0, 1, 2, \dots$$

Since all  $w^{\mu_n}$  fix  $0, 1, \infty$ , it follows that  $0, 1, \infty \in P_{f,n}$ .

**Definition 5** (Spherical Version). *We say  $f$  has bounded geometry if there is a constant  $b > 0$  and a point  $\tau_0 \in T_f$  such that*

$$d_{sp}(p_n, q_n) \geq b$$

for  $p_n, q_n \in P_{f,n}$  and  $n \geq 0$ . Here

$$d_{sp}(z, z') = \frac{|z - z'|}{\sqrt{1 + |z|^2} \sqrt{1 + |z'|^2}}$$

is the spherical distance on  $\hat{\mathbb{C}}$ .

Note that  $d_{sp}(z, \infty) = \frac{|z|}{\sqrt{1 + |z|^2}}$ . Away from infinity the spherical metric and Euclidean metric are equivalent. Precisely, for any bounded  $S \subset \mathbb{C}$ , there is a constant  $C > 0$  which depends only on  $S$  such that

$$C^{-1}d_{sp}(x, y) \leq |x - y| \leq Cd_{sp}(x, y) \quad \forall x, y \in S.$$

Consider the hyperbolic Riemann surface  $R = \hat{\mathbb{C}} \setminus P_f$  equipped with the standard complex structure as the basepoint  $[0] \in T_f$ . A point  $\tau$  in  $T_f$  defines another complex structure  $\tau$  on  $R$ . Denote by  $R_\tau$  the hyperbolic Riemann surface  $R$  equipped with the complex structure  $\tau$ .

A simple closed curve  $\gamma \subset R$  is called *non-peripheral* if each component of  $\hat{\mathbb{C}} \setminus \gamma$  contains at least two points of  $P_f$ . Let  $\gamma$  be a non-peripheral simple closed curve in  $R$ . For any  $\tau = [\mu] \in T_f$ , let  $l_\tau(\gamma)$  be the hyperbolic length of the unique closed geodesic homotopic to  $\gamma$  in  $R_\tau$ . The bounded geometry property can be stated in terms of hyperbolic geometry as follows.

**Definition 6** (Hyperbolic version). *We say  $f$  has bounded geometry if there is a constant  $a > 0$  and a point  $\tau_0 \in T_f$  such that  $l_{\tau_n}(\gamma) \geq a$  for all  $n \geq 0$  and all non-peripheral simple closed curves  $\gamma$  in  $R$ .*

The above definitions of bounded geometry are equivalent because of the following lemma and the fact that we have normalized so that  $0, 1, \infty$  always belong to  $P_f$ .

**Lemma 4.** *Consider the hyperbolic Riemann surface  $\hat{\mathbb{C}} \setminus X$ , where  $X$  is a finite subset of  $\hat{\mathbb{C}}$  such that  $0, 1, \infty \in X$ , equipped with the standard complex structure. Let  $a > 0$  be a constant. If every simple closed geodesic in  $\hat{\mathbb{C}} \setminus X$  has hyperbolic length greater than  $a$ , then the spherical distance between any two distinct points in  $X$  is bounded below by a bound  $b > 0$  which depends only on  $a$  and  $m = \#(X)$ .*

*Proof.* If  $m = 3$  there are no non-peripheral simple closed curves so in the following argument we may assume that  $m \geq 4$ . Let  $X = \{x_1, \dots, x_{m-1}, x_m = \infty\}$  and let  $|\cdot|$  denote the Euclidean metric on  $\mathbb{C}$ .

Suppose  $0 = |x_1| \leq \dots \leq |x_{m-1}|$ . Let  $M = |x_{m-1}|$ . Then  $|x_2| \leq 1$ , and we have

$$\prod_{2 \leq i \leq m-2} \frac{|x_{i+1}|}{|x_i|} = \frac{|x_{m-1}|}{|x_2|} \geq M.$$

Hence

$$\max_{2 \leq i \leq m-2} \left\{ \frac{|x_{i+1}|}{|x_i|} \right\} \geq M^{\frac{1}{m-3}}.$$

Let

$$A_i = \{z \in \mathbb{C} \mid |x_i| < z < |x_{i+1}|\}$$

and let  $\text{mod}(A_i) = \frac{1}{2\pi} \log \frac{|x_{i+1}|}{|x_i|}$  be its modulus. Then for some integer  $2 \leq i_0 \leq m-2$  it follows that

$$\text{mod}(A_{i_0}) \geq \frac{\log M}{2\pi(m-3)}.$$

Denote the extremal length of the core curve  $\gamma_{i_0}$  in  $A_{i_0} \subset \hat{\mathbb{C}} \setminus X$  by  $\|\gamma_{i_0}\|$ . By properties of extremal length,

$$\|\gamma_{i_0}\| = \frac{1}{\text{mod}(A_{i_0})} \leq \frac{2\pi(m-3)}{\log M}.$$

Since extremal length is defined by taking a supremum over all metrics and the area of  $\hat{\mathbb{C}} \setminus X$  which is  $2\pi(m-2)$  for every hyperbolic metric,

$$\|\gamma_{i_0}\| \geq \frac{l_{\tau_n}^2(\gamma)}{2\pi(m-2)} \geq \frac{a^2}{2\pi(m-2)}.$$

Setting  $a' = \frac{a^2}{2\pi(m-2)}$ , these inequalities imply

$$M \leq M_0 = e^{\frac{2\pi(m-3)}{a'}}.$$

Thus the spherical distance between  $\infty$  and any finite point in  $X$  has a positive lower bound  $b$  which depends only on  $a$  and  $m$ .

Next we show that the spherical distance between any two finite points in  $X$  has a positive lower bound depending only on  $a$  and  $m$ . By the equivalence of the spherical and Euclidean metrics in a bounded set in the plane, it suffices to prove that  $|x - y|$  is greater than a constant  $b$  for any two finite points in  $X$ .

First consider the map  $\alpha(z) = 1/z$  which is a hyperbolic isometry from  $X$  to  $\alpha(X)$ . It preserves the set  $\{0, 1, \infty\}$  so that  $0, 1, \infty \in \alpha(X)$ . For any  $2 \leq i \leq m-1$ , the above argument implies that  $1/|x_i| \leq M_0$  and hence  $|x_i| \geq 1/M_0$ . Similarly, for any  $x_i \in X$ ,  $2 \leq i \leq m-1$ , consider the map  $\beta(z) = z/(z - x_i)$ . It maps  $\{0, \infty, x_i\}$  to  $\{0, 1, \infty\}$  so that  $\beta(X)$  contains  $\{0, 1, \infty\}$  and it is also a hyperbolic isometry. For any  $2 \leq i \leq m-1$ , the above argument implies that  $|x_j|/|x_j - x_i| \leq M_0$  which in turn implies that  $|x_j - x_i| \geq 1/M_0^2$  proving the lemma.  $\square$

## 9. COMPACTNESS

The normalized functions in  $\mathcal{E}_{p,q}$  are determined by the  $p + q + 1$  coefficients of the polynomials  $P$  and  $Q$ . This identification defines an embedding into  $\mathbb{C}^{p+q+1}$  and hence a topology on  $\mathcal{E}_{p,q}$ .

Given  $f \in \mathcal{TE}_{p,q}$  and given any  $\tau_0 = [\mu_0] \in T_f$ , let  $\tau_n = \sigma^n(\tau_0) = [\mu_n]$  be the sequence generated by  $\sigma$ . Let  $w^{\mu_n}$  be the normalized quasiconformal map with Beltrami coefficient  $\mu_n$ . Then

$$E_n = w^{\mu_n} \circ f \circ (w^{\mu_{n+1}})^{-1} \in \mathcal{E}_{p,q}$$

since it preserves  $\mu_0$  and hence is holomorphic. This gives a sequence of maps in  $\mathcal{E}_{p,q}$  and a sequence of subsets  $P_{f,n} = w^{\mu_n}(P_f)$ . Note that  $P_{f,n}$  is not, in general, the post-singular set  $P_{E_n}$  of  $E_n$ . If it were, we would have a fixed point of  $\sigma$ .

Recall that  $\Omega_f$  denotes the set of critical points of  $f$  and that

$$S_f = f(\Omega_f) \cup \{0, \infty\}$$

denotes the set of singular values of  $f$ , that is the critical values and asymptotic values of  $f$ . Then

$$\Omega_{E_n} = w^{\mu_{n+1}}(\Omega_f)$$

is the set of critical points of  $E_n$  and

$$S_{E_n} = E_n(\Omega_{E_n}) \cup \{0, \infty\} = w^{\mu_n}(S_f)$$

is the set of singular values of  $E_n$ . One can check that

$$0, 1, \infty \in S_{E_n} \subset P_{f,n}$$

and

$$E_n(1) = w^{\mu_n}(f(1)) \in P_{f,n}.$$

That is, all singular values are in the set  $P_{f,n}$  for which we posited the bounded geometry condition in section 8. In this section we will use this fact about  $S_{E_n}$  and the bounded geometry condition on  $\{P_{f,n}\}$  to deduce that the sequence  $\{E_n\}$  for a post-singularly finite  $f \in \mathcal{TE}_{p,q}$  is compact. As we pointed out in the previous section, to avoid a separate discussion of the simplest case, we always assume that  $\#(P_f) > 3$ . We will use this compactness condition in section 10.

From a conceptual point of view, the compactness condition is very natural and simple. From a technical point of view, however, it is not at all obvious. To present the ideas clearly, we divide our proof into two parts. We first give the slightly simpler



proof for  $f \in \mathcal{TE}_{0,q}$  with  $f(1) \neq 1$  and then give a detailed argument in the general case  $f \in \mathcal{TE}_{p,q}$ .

Because we are interested in compactness, the arguments all involve taking subsequences. For readability, we always tacitly assume we work with an appropriate subsequence in the following arguments and, without saying so, use the same index  $n$  for the subsequence.

We begin with a simple lemma about polynomials. We leave the proof to the reader as an exercise.

**Lemma 5.** *Let  $f$  be a polynomial of degree  $d$ . If every critical point of  $f$  is also a root of  $f$ , then  $f(z) = a(z - c)^d$  for some constants  $a$  and  $c$ .*

The next lemma is interesting as a result in the theory of polynomials. We have a family of polynomials  $f_n$  and are interested in what happens to the roots of subsequences when the coefficients tend to zero or infinity. In the lemma we talk about “sequences of roots” so we need to have a way to keep track of which root is which and thus we assume the roots are labelled by decreasing absolute value. In the application, where the polynomials come from iterating the Thurston map, keeping track is not a problem because once we label the roots and critical points of the given  $f$ , the labels of the corresponding points for  $E_n$  are determined by  $w^{\mu_n}$ .

**Lemma 6.** *Let  $d > 0$  be an integer and  $f_n(z) = a_{d,n}z^d + \dots + a_{1,n}z + a_{0,n}$  be a family of polynomials of degree  $d$ . Then, passing to a subsequence, one of the following must occur:*

- (i)  $f_n$  converges to a constant.
- (ii)  $f_n$  converges to a non-constant polynomial of degree less than or equal to  $d$ . Set  $\lim a_{d,n} = a$ . If  $a \neq 0$  the degree of the limit polynomial  $f$  is  $d$ ; if  $a = 0$  the degree of the limit is strictly less than  $d$  and there exists a sequence of roots  $\xi_n$  of  $f_n$  such that  $\xi_n \rightarrow \infty$ . If there are  $k \leq d$  sequences of roots  $\xi_{j,n}$ ,  $j = 1, \dots, k$  with  $\xi_{j,n} \rightarrow \infty$ , then the first  $k$  coefficients of the limit polynomial vanish.
- (iii)  $f_n$  converges to  $\infty$  except for at most  $d$  points.

*Proof.* Let  $M_n = \max\{|a_{d,n}|, \dots, |a_{0,n}|\}$ . If  $M_n \rightarrow 0$ , then on any compact subset of  $\mathbb{C}$ ,  $f_n$  converges uniformly to 0.

If  $M_n$  is bounded but does not tend to 0 and if  $\lim a_{d,n} = a \neq 0$ ,  $f_n$  converges to a non-constant polynomial of degree  $d$ . Such a polynomial has  $d$  roots counted with multiplicity. Assume that 0 is not a root of  $f_n$ ; if it is, write  $f_n = z^\nu \hat{f}_n(z)$  and proceed with the argument for  $\hat{f}_n$ .

If  $a = 0$ , then the degree of the limit polynomial is less than  $d$ . Write  $f_n = a_{d,n}g_n$  where  $g_n$  is a monic polynomial whose coefficients are symmetric functions of the  $d$  (not necessarily distinct) roots  $\xi_{1,n}, \dots, \xi_{d,n}$ . Write

$$(2) \quad g_n(z) = (z - \xi_{1,n}) \cdots (z - \xi_{d,n}) = z^d - (\xi_{1,n} + \cdots + \xi_{d,n})z^{d-1} + \cdots + (-1)^d \xi_{1,n} \cdots \xi_{d,n}.$$

We see that if all the roots remain bounded, the coefficients of  $g_n$  remain bounded. Since by assumption  $a = 0$ , it follows that  $M_n \rightarrow 0$  contradicting the assumption that  $\lim M_n \neq 0$ . Therefore there is at least one sequence of roots  $\xi_{i,n}$  that converges to infinity. Suppose the roots are labeled in decreasing order of magnitude and that the first  $k$  roots  $\xi_{1,n}, \dots, \xi_{k,n}$  converge to infinity and the others remain finite. The product  $a_{d,n}\xi_{1,n} \dots \xi_{k,n}$  is one of the summands in  $a_{d-k,n}$ , so if it tends to zero, by the ordering of the roots, it follows that the whole coefficient tends zero. Now consider the remaining coefficients  $a_{d-j,n}, j > k$ . Although their summands contain products of  $d-j > d-k$  roots, they must also tend to zero. This is because  $a_{d,n}$  is tending to zero and any roots in the product that are not among the first  $k$  are finite. This contradicts  $\lim M_n \neq 0$ . Therefore  $|\lim a_{d,n}\xi_{1,n} \dots \xi_{k,n}|$  has a non-zero limit there exists a  $C > 0$  such that

$$C^{-1} < |a_{d,n}\xi_{1,n} \dots \xi_{k,n}| < C.$$

For the coefficients,  $a_{d-j,n}, j < k$ , the summands each contain a proper subset of the factors  $a_{d,n}\xi_{1,n}, \dots, \xi_{k,n}$  and these all have limit 0.

If  $M_n$  is unbounded write  $f_n = M_n h_n$ . From the last paragraph, we know that  $h_n$  converges to a polynomial  $h$ . If the degree of  $h$  is less than  $d$ , then there exists a sequence of roots  $\xi_{i,n}$  of  $h_n$  (and hence  $f_n$ ) such that  $\xi_{i,n} \rightarrow \infty$ . If  $z$  is any point that is not a root of the limit polynomial, that is,  $h(z) \neq 0$ , then  $f_n(z) = M_n h_n(z)$  converges to infinity. Finally, if  $h$  has degree  $d$ , then  $\lim a_{d,n} = \infty$ .  $\square$

**Theorem 2.** *Given an  $f \in \mathcal{T}E_{0,q}$  with bounded geometry and  $f(1) \neq 1$ , then the sequence  $E_n$  generated by Thurston iteration is contained in a compact subset of  $\mathcal{E}_{0,q}$ .*

*Proof.* In this case,

$$E_n(z) = e^{Q_n(z)}.$$

By our normalization  $Q_n(0) = 0$ . Therefore

$$Q_n(z) = b_{q,n}z^q + \dots + b_{1,n}z, \quad b_{q,n} \neq 0$$

We will prove that  $Q_n$  is contained in a compact subset of the space of all polynomials of degree  $q$ .

Let  $M_n = \max\{|b_{q,n}|, \dots, |b_{1,n}|\}$ . If passing to a subsequence,  $M_n \rightarrow 0$ , there exists a subsequence of  $Q_n$  which converges to 0 on any compact subset of  $\mathbb{C}$ . This implies that the sequence  $E_n(1) \neq 1$  in  $P_{f,n}$  tends to 1 as  $n \rightarrow \infty$ , which contradicts the bounded geometry condition. This contradiction shows that the sequence  $M_n$  is bounded away from 0.

In the case that  $M_n$  is bounded,  $Q_n$  has a convergent subsequence with a limit polynomial  $Q$  of degree less or equal to  $q$ . From the previous paragraph, we know that  $\lim M_n = M > 0$ . If  $\deg Q = q$ , it belongs to  $\mathcal{E}_{0,q}$  and  $Q_n$  is in a compact subset of  $\mathcal{E}_{0,q}$  as claimed.

Now we claim that if  $\lim M_n = M > 0$  and  $f$  has bounded geometry then  $\deg Q = q$ . We prove the claim by contradiction. Suppose  $1 \leq \deg Q < q$ . Let  $Q_n = b_{q,n}g_n$ ,

where  $b_{q,n}$  is the leading coefficient of  $Q_n$ . Then  $b_{q,n} \rightarrow 0$ . Suppose the critical points of  $g_n$  are  $\xi_{1,n}, \dots, \xi_{q-1,n}$ . We have

$$g'_n(z) = q(z - \xi_{1,n}) \cdots (z - \xi_{q-1,n}) = q(z^{q-1} - (\xi_{1,n} + \cdots + \xi_{q-1,n})z^{q-2} + \cdots + (-1)^{q-1}\xi_{1,n} \cdots \xi_{q-1,n}).$$

Then

$$g_n(z) = z^q - \frac{q}{q-1}(\xi_{1,n} + \cdots + \xi_{q-1,n})z^{q-1} + \cdots + (-1)^{q-1}q\xi_{1,n} \cdots \xi_{q-1,n}z.$$

Thus

$$b_{q-1,n} = -b_{q,n} \frac{q}{q-1}(\xi_{1,n} + \cdots + \xi_{q-1,n}), \dots, b_{1,n} = b_{q,n}(-1)^{q-1}q\xi_{1,n} \cdots \xi_{q-1,n}.$$

By lemma 6, there is at least one sequence of critical points  $\xi_{j,n}$  going to  $\infty$  as  $n \rightarrow \infty$ . Suppose the sequences are ordered by decreasing magnitude and only the first  $k$ ,  $1 \leq k \leq q-1$ , sequences of critical points  $\xi_{1,n}, \dots, \xi_{k,n}$  tend to  $\infty$  as  $n \rightarrow \infty$ . Note that the sequences may tend to infinity at different rates so that the first  $l \leq k$  sequences grow at the same (fastest) rate.

Since the sequence  $M_n$  is bounded and bounded away from 0, by the proof of lemma 6, there is a constant  $C > 0$  such that

$$(3) \quad C^{-1}|\xi_{1,n} \cdots \xi_{k,n}|^{-1} \leq |b_{q,n}| \leq C|\xi_{1,n} \cdots \xi_{k,n}|^{-1}.$$

Furthermore, since the  $\xi_{i,n}$  are ordered by decreasing magnitude, there is another constant,  $C_1 > 0$ , such that

$$(4) \quad C_1^{-1} \leq \frac{|\xi_{i,n}|}{|\xi_{1,n}|} \leq C_1, \quad \forall \ 1 \leq i \leq l$$

and

$$(5) \quad \frac{|\xi_{j,n}|}{|\xi_{1,n}|} \rightarrow 0, \quad \forall l < j \leq q-1.$$

Consider any critical point  $\xi_{i,n}$ ,  $1 \leq i \leq l$ . Then

$$Q_n(\xi_{i,n}) = b_{q,n}g_n(\xi_{i,n}) = b_{q,n}\xi_{i,n}^q \frac{g_n(\xi_{i,n})}{\xi_{i,n}^q}.$$

By (3), (4) and (5),  $b_{q,n}\xi_{i,n}^q \rightarrow \infty$  so that if the critical value  $Q_n(\xi_{i,n})$  were bounded, it would follow that

$$\frac{g_n(\xi_{i,n})}{\xi_{i,n}^q}$$

tended to 0. Let

$$c_i = \lim_{n \rightarrow \infty} \frac{\xi_{i,n}}{\xi_{1,n}}.$$

By our assumptions about the sequences  $\xi_{i,n}$ , it follows that  $c_i \neq 0$ ,  $1 \leq i \leq l$ ,  $c_1 = 1$  and  $c_i = 0$ ,  $l < i \leq q-1$ . Thus

$$\begin{aligned}
 (6) \quad \lim_{n \rightarrow \infty} \frac{g_n(\xi_{i,n})}{\xi_{i,n}^q} &= \lim_{n \rightarrow \infty} \frac{\int_0^{\xi_{i,n}} q(z - \xi_{1,n}) \cdots (z - \xi_{q-1,n}) dz}{\xi_{i,n}^q} \\
 &= \lim_{n \rightarrow \infty} \frac{\int_0^{\frac{\xi_{i,n}}{\xi_{1,n}}} q(\xi_{1,n}z - \xi_{1,n}) \cdots (\xi_{1,n}z - \xi_{q-1,n}) d(\xi_{1,n}z)}{\xi_{i,n}^q} \\
 &= \frac{\int_0^{c_i} \zeta^{q-l-1} (\zeta - c_1) \cdots (\zeta - c_l) d\zeta}{c_i^q} = 0
 \end{aligned}$$

Set  $F(z) = \int_0^z \zeta^{q-l-1} (\zeta - c_1) \cdots (\zeta - c_l) d\zeta$ . Then  $F$  is a polynomial of degree  $q$  and its critical points are 0, with multiplicity  $q-l-1$ , and  $c_i$ ,  $1 \leq i \leq l$ . Moreover by equation (6), its zeros are  $c_i$ ,  $1 \leq i \leq l$ , and 0 with multiplicity  $q-l$ . By lemma 5, this implies all the  $c_i = 0$  which is a contradiction. Therefore there exists an  $i$ ,  $1 \leq i \leq l$ , such that  $Q_n(\xi_{i,n}) \rightarrow \infty$  as  $n \rightarrow \infty$ .

There are two ways that  $Q_n(\xi_{i,n})$  can tend to  $\infty$  as  $n \rightarrow \infty$ : (1) the values  $Q_n(\xi_{i,n})$  tend to  $\infty$  and remain inside asymptotic tracts, or (2) the values of  $Q_n(\xi_{i,n})$  tend to  $\infty$  asymptotic to a Julia direction. This is a dichotomy that comes up over and over again. In the first case, the critical values  $E_n(\xi_{i,n}) = e^{Q_n(\xi_{i,n})} \in P_{f,n}$  tend to either 0 or  $\infty$  as  $n \rightarrow \infty$  depending on which asymptotic tract the subsequence  $Q_n(\xi_{i,n})$  lies in. This contradicts the bounded geometry condition. In the second case, the critical values  $E_n(\xi_{i,n}) = e^{Q_n(\xi_{i,n})} \in P_{f,n}$  remain bounded but one of the winding numbers in the topological constraint  $\eta_{E_n}$  as defined in section 4 increases to  $\infty$ . Since  $\eta_{E_n} = \eta_f$ , this is a contradiction so that  $\deg g = q$  proving our claim.

Now we claim  $M_n$  unbounded also implies  $\deg g = q$ . Suppose there is a subsequence of  $M_n$  converging to  $\infty$ . By our notational convention,  $M_n \rightarrow \infty$  as  $n \rightarrow \infty$ . Consider  $g_n = Q_n/M_n$ . Then the coefficients of  $g_n$  remain bounded and  $g_n$  converges to a nonconstant polynomial  $g \not\equiv 0$ . By lemma 6 and the discussion above, if the degree of  $g$  were less than  $q$ , there would exist a sequence of critical points  $\xi_{i,n}$  of  $g_n$  such that  $g_n(\xi_{i,n}) \rightarrow \infty$  and hence that  $Q_n(\xi_{i,n}) = M_n g_n(\xi_{i,n}) \rightarrow \infty$ . We have the same dichotomy again, and, as above, in both cases either bounded geometry or the topological constraint is violated so that  $\deg g = q$ .

Finally we claim that the assumptions  $M_n$  is unbounded and  $\deg g = q$  also lead to a contradiction. In this case, by lemma 6, all the critical points of  $g_n$  are bounded and we can find convergent subsequences  $\xi_{i,n}$ . Let  $c_i = \lim_{n \rightarrow \infty} \xi_{i,n}$ ; then both  $c_i$  and  $\lim_{n \rightarrow \infty} g_n(\xi_{i,n}) = g(c_i)$  are bounded. The bounded geometry condition implies that all the critical values of  $Q_n(\xi_{i,n})$  are bounded and bounded away from 0. Since  $Q_n(\xi_{i,n}) = M_n g_n(\xi_{i,n})$ , it follows that  $\lim_{n \rightarrow \infty} g_n(\xi_{i,n}) = g(c_i) = 0$  for all  $1 \leq i \leq q-1$  so that  $c_i$ ,  $i = 1, \dots, q$ , are both critical points and roots of  $g$ . Lemma 5 implies that  $g(z) = a(z-c)^q$ . But  $g(0) = \lim_{n \rightarrow \infty} Q_n(0) = 0$  so that  $c = c_i = 0$  for all  $i$ .

This further implies that

$$\lim_{n \rightarrow \infty} \frac{b_{i,n}}{b_{q,n}} = 0, \quad \forall \quad 1 \leq i \leq q-1 \quad \text{and} \quad \lim_{n \rightarrow \infty} |b_{q,n}| = \lim_{n \rightarrow \infty} M_n = \infty.$$

Now consider the sequence

$$E_n(1) = e^{Q_n(1)} = e^{b_{q,n}(1 + \frac{b_{q-1,n}}{b_{q,n}} + \dots + \frac{b_{1,n}}{b_{q,n}})}.$$

As above there are two ways that  $b_{q,n}$  and hence  $Q_n(1)$  can tend to  $\infty$  as  $n \rightarrow \infty$ : (1) the values  $Q_n(1)$  tend to  $\infty$  and remain inside the asymptotic tracts, or (2) they approach  $\infty$  asymptotic to a Julia direction.

In the first case, the critical value  $E_n(1) \in P_{f,n}$  tends to 0 or  $\infty$  as  $n \rightarrow \infty$  depending on which asymptotic tract the subsequence lies in contradicting the bounded geometry condition. In the second case, the critical value  $E_n(1) \in P_{f,n}$  remains bounded but at least one of the winding numbers in the topological constraint  $\eta_{E_n}$  of section 4 increases to  $\infty$ . This contradiction completes the proof of the theorem.  $\square$

The ideas in the general case where the degree of the polynomial  $P$  in  $E$  is positive, are basically the same but the techniques for dealing with polynomials and critical points of  $E$  are more complicated. We start by investigating critical points and critical values of a sequence of polynomials  $f_n$  of fixed degree  $d > 0$ . Suppose

$$f_n(z) = a_{d,n}z^d + \dots + a_{1,n}z + a_{0,n}, \quad a_{d,n} \neq 0, \quad n = 1, 2, \dots.$$

Let  $\xi_{i,n}$  ( $i = 1, \dots, d-1$ ) be the critical points of  $f_n$  labelled by decreasing order of magnitude and set

$$M_n = \max_{1 \leq i \leq d-1} \{|a_{i,n}|\}.$$

By passing to a subsequence, we assume that  $\lim_{n \rightarrow \infty} M_n = M$ . There are three possibilities:

- (1)  $M = 0$ ,
- (2)  $0 < M < \infty$ ,
- (3)  $M = \infty$ .

In case (1), part (1) of Lemma 6 implies that  $f_n \rightarrow 0$  uniformly on any compact subset in  $\mathbb{C}$ . We deal with case (3) in the following lemma.

**Lemma 7.** *Suppose  $M = \infty$  and  $f_n(0) = 0$ , then either*

- (i) *there exists a sequence of critical points  $\xi_{i,n}$  such that  $f_n(\xi_{i,n}) \rightarrow \infty$  or*
- (ii)  *$f_n(z) = a_{d,n}z^d$  and  $\lim_n a_{d,n} = \infty$ .*

*Proof.* Consider  $g_n = f_n/M_n$  so that  $g_n$  converges to a nonconstant polynomial  $g$ . If the degree of  $g$  is less than  $d$ , the degree of  $g'$  is less than  $d-1$  and lemma 6 implies that there exists a sequence of critical points  $\xi_{i,n}$  tending to infinity. The argument of theorem 2 showing that if the critical points of  $Q_n$  tend to infinity, the corresponding critical values do applied here shows that  $g_n(\xi_{i,n}) \rightarrow \infty$ . This in turn implies that  $f_n(\xi_{i,n}) = M_n g_n(\xi_{i,n}) \rightarrow \infty$  as  $n \rightarrow \infty$ .

If the degree of  $g$  is equal to  $d$ , then all the sequences of critical points  $\xi_{i,n}$  and the corresponding sequences of critical values  $g_n(\xi_{i,n})$  are bounded. Let  $c_i = \lim_{n \rightarrow \infty} \xi_{i,n}$  so that  $c_i$ ,  $i = 1, \dots, d-1$ , are the critical points of  $g$  and  $\lim_{n \rightarrow \infty} g_n(\xi_{i,n}) = g(c_i)$  are the corresponding critical values. If all the sequences of critical values  $f_n(\xi_{i,n})$  are bounded, since  $f_n(\xi_{i,n}) = M_n g_n(\xi_{i,n})$  and  $M = \infty$ , we conclude that  $g(c_i) = 0$  for all  $1 \leq i \leq d-1$ . Now lemma 5 implies that  $g(z) = a(z-c)^d$ , and since  $g(0) = 0$ ,  $c = c_i = 0$ . This further implies the assertions of the lemma

$$f_n(z) = a_{d,n} z^d \text{ and } \lim_n a_{d,n} = \infty.$$

□

Now we consider case (2). In this case, all the coefficients  $a_{i,n}$  remain bounded and a subsequence of  $f_n$  converges to some  $f$ . If  $\deg f = d$ , then all the critical points  $\xi_{i,n}$  are bounded and converge to the critical points of  $f$ .

If  $\deg f < d$ , there is at least one sequence  $\xi_{i,n}$  that tends to  $\infty$  as  $n \rightarrow \infty$ ; we may assume the critical points are labeled in decreasing order of magnitude so that the first  $k$ ,  $1 \leq k \leq d-1$ , points  $\xi_{1,n}, \dots, \xi_{k,n}$  go to infinity and the remaining ones stay bounded. We may also assume the first  $l$  critical points,  $1 \leq l \leq k$ ,  $\xi_{1,n}, \dots, \xi_{l,n}$  grow fastest. Taking appropriate subsequences, this means

$$\lim_{n \rightarrow \infty} \frac{|\xi_{i,n}|}{|\xi_{1,n}|} = c_i, \quad \forall 1 \leq i \leq l, \quad 0 < c_i < \infty.$$

and

$$\lim_{n \rightarrow \infty} \frac{|\xi_{j,n}|}{|\xi_{1,n}|} = 0, \quad \forall l+1 \leq j \leq k.$$

In addition there is a constant  $C > 0$  such that

$$|\xi_{s,n}| \leq C, \quad \forall k+1 \leq s \leq d-1.$$

We will need to consider the following situation. Let  $\zeta_n \in \mathbb{C}$  be a sequence of non-roots converging to infinity. What can we say about  $\lim f_n(\zeta_n)$ ? Consider the following examples:

$$(7) \quad f_{1,n}(z) = \frac{1}{n^4}(z - (n+1)^2)(z - (n+2))(z - (n+3))$$

$$(8) \quad f_{2,n}(z) = \frac{1}{n^4}(z - (n+1)^2)(z - (n+2))(z - (n+3))(z - 1)$$

$$(9) \quad f_{3,n}(z) = \frac{1}{n^4}(z - (n+1)^2)(z - (n+2))(z - (n+3))(z - 1)(z - 2)$$

$$(10) \quad f_{4,n}(z) = \frac{1}{n^3}(z - (n+1))(z - (n+2))(z - (n+3))(z - 1)(z - 2)(z - 3)$$

The limit polynomials are respectively,  $f_1 \equiv -1$ ,  $f_2 = -(z-1)$ ,  $f_3 = -(z-1)(z-2)$ , and  $f_4 = -(z-1)(z-2)(z-3)$ . Note that there are sequences  $\zeta_n$  of non-roots converging to infinity for which we cannot interchange the limits; that is, for all  $j$ ,  $\lim f_{j,n}(\zeta_n) \neq f_j(\lim \zeta_n)$ .

In each case there is a sequence  $\zeta_n$  closely following, say, the root  $n + 2$  of  $f_{j,n}$  such that  $\lim_n(f_{j,n}(\zeta_n)) \neq f_j(n)$ . We have

$$\begin{aligned}\lim f_{1,n}(n) &= 0 \\ \lim f_{2,n}(n) &= 0 \\ \lim f_{3,n}(n) &= -6 \\ \lim f_{4,n}(n) &= -6\end{aligned}$$

These examples illustrate that the behavior of  $f_n(\zeta_n)$  depends delicately on the number of roots that go to infinity, how fast they go to infinity, and which roots  $\zeta_n$  is asymptotic to. In the next propositions we address this problem in the situations where we will want to apply it.

**Proposition 2.** *Suppose that we have a sequence of polynomials  $f_n$  of fixed degree  $d$  converging to a polynomial of lower degree but not identically zero. Write the roots as  $\xi_{1,n}, \dots, \xi_{d,n}$  and assume the first  $k$  of these roots go to infinity, but the others stay bounded.*

*Assume that there is a sequence of non-roots  $\zeta_n$  converging to infinity closely following each of the roots that tends to infinity. That is, for  $j = 1, \dots, k$ ,  $\zeta_n - \xi_{j,n} = \alpha_{j,n}$  and  $\alpha_j = \lim \alpha_{j,n}$  is a finite non-zero constant. Then, if  $\lim f_n(\zeta_n) \neq 0, \infty$ , we have  $k = d/2$  and for all  $j = 1, \dots, k$ ,  $\lim \xi_{j,n} a_{d,n}^{1/k}$  is finite and non-zero.*

*Proof.* Write  $f_n(\zeta_n) = a_{d,n} \prod_{j=1}^d (\zeta_n - \xi_{j,n})$  and assume first none of the roots are zero. Since the limit polynomial is not zero,  $\lim a_{d,n} \xi_{1,n} \dots \xi_{d,n} \neq 0$  and as in lemma 6,  $\lim a_{d,n} \xi_{1,n} \dots \xi_{k,n} \neq 0$ . Write  $f_n$  as follows:

$$(11) \quad \frac{f_n(\zeta_n)}{a_{d,n} \xi_{1,n} \dots \xi_{k,n}} = \alpha_{1,n} \dots \alpha_{k,n} \frac{(\zeta_n - \xi_{k+1,n}) \dots (\zeta_n - \xi_{d,n})}{\xi_{1,n} \dots \xi_{k,n}}.$$

By hypothesis, for any  $j = 1, \dots, k$ ,  $\zeta_n$  is comparable to each of the first  $k$  roots; that is  $\lim |\zeta_n / \xi_{j,n}|$  is finite and non-zero  $1 \leq j \leq k$ . Since the remaining roots stay bounded,  $\lim |\zeta_n / \xi_{j,n}| = \infty$ ,  $i = k + 1, \dots, d$ . Suppose first  $k < d/2$ . We rewrite equation (11) as

$$\frac{f_n(\zeta_n)}{a_{d,n} \xi_{1,n} \dots \xi_{k,n}} = \alpha_{1,n} \dots \alpha_{k,n} \left( \frac{\zeta_n - \xi_{k+1,n}}{\xi_{1,n}} \right) \dots \left( \frac{\zeta_n - \xi_{2k}}{\xi_{k,n}} \right) (\zeta_n - \xi_{2k+1}) \dots (\zeta_n - \xi_{d,n}).$$

The first  $2k$  factors have finite non-zero limits and the rest converge to  $\infty$  so the limit of the product is  $\infty$ . Now suppose  $k > d/2$ . Rewriting again we have

$$\frac{f_n(\zeta_n)}{a_{d,n} \xi_{1,n} \dots \xi_{k,n}} = \alpha_{1,n} \dots \alpha_{k,n} \frac{(\zeta_n - \xi_{k+1,n}) \dots (\zeta_n - \xi_{d,n})}{\xi_{1,n} \dots \xi_{d-k,n}} \left( \frac{1}{\xi_{d-k-1,n} \dots \xi_{d,n}} \right)$$

Now the first  $2k - d$  factors have finite non-zero limits and the last converges to 0 so the limit of the product is 0. It follows that  $k = d/2$ .

By hypothesis there are constants such that

$$0 < C^{-k} < \left| \frac{\xi_{1,n} \cdots \xi_{k,n}}{\xi_{1,n}^k} \right| < C^k \text{ for all } n$$

which, because  $\xi_{k+1,n} \cdots \xi_{d,n}$  are finite, implies that  $\lim a_{d,n} \xi_{1,n}^k$  and  $\lim a_{d,n} \xi_{j,n}^k$  for  $j = 1, \dots, k$  are finite and non-zero.

If zero is a root of  $f$  of multiplicity  $\nu > 0$ , write  $f = z^\nu g$  and apply the above to  $g$ .  $\square$

In the above proposition we assumed that the roots that tend to infinity all grow at comparable rates. The sequence of functions  $f_{4,n}$  illustrate this proposition. In the next proposition, some of the roots tend to infinity faster than others. It is illustrated by the functions  $f_{3,n}$ .

**Proposition 3.** *Suppose that we have a sequence of polynomials  $f_n$  of fixed degree  $d$  converging to a polynomial of lower degree but not identically zero. Write the roots as  $\xi_{1,n}, \dots, \xi_{d,n}$  and assume the first  $k$  of these roots go to infinity and that they are labelled in decreasing order of magnitude. Assume that there is a sequence of non-roots  $\zeta_n$  converging to infinity with the property that for the first  $r$  of the roots,  $j = 1, \dots, r$ ,  $\lim \zeta_n / \xi_{j,n} = 0$ ; for the next  $s$  roots,  $j = r+1, \dots, r+s$ , there are finite non-zero constants  $\alpha_j$  such that  $\lim(\zeta_n - \xi_{j,n}) = \alpha_j$ ; and, for the remaining  $k - r - s$  roots tending to infinity,  $j = r+s+1, \dots, k$ ,  $\lim \zeta_n / \xi_{j,n} = \infty$ . Assume that  $\lim f_n(\zeta_n) \neq 0, \infty$ .*

*Next assume that there is another sequence of non-roots  $\eta_n$  converging to infinity faster than  $\zeta_n$  and asymptotic the roots  $\xi_{i,n}$ ,  $i = t, \dots, r$ ,  $t \geq 1$ ; that is,  $\lim |\eta_n / \zeta_n| = \infty$  and for each  $i = t, \dots, r$ ,  $\lim \eta_n - \xi_{i,n} = \beta_i$  is finite and non-zero. Then  $\lim f_n(\eta_n) = \infty$ . Similarly, if  $\eta_n$  converges to infinity slower than  $\zeta_n$ , and is asymptotic to  $\xi_{i,n}$ ,  $i = t_1, \dots, t_2$ ,  $r+s+1 \leq t_1 \leq t_2 \leq k$ , then  $\lim f_n(\eta_n) = 0$ .*

*Proof.* Again, write  $f_n$  as

$$\begin{aligned} \frac{f_n(\zeta_n)}{a_{d,n} \xi_{1,n} \cdots \xi_{k,n}} &= \left( \frac{\zeta_n}{\xi_{1,n}} - 1 \right) \cdots \left( \frac{\zeta_n}{\xi_{r,n}} - 1 \right) \alpha_{r+1,n} \cdots \alpha_{r+s,n} \\ &\quad \left( \frac{\zeta_n}{\xi_{r+s+1,n}} - 1 \right) \cdots \left( \frac{\zeta_n}{\xi_{k,n}} - 1 \right) \frac{(\zeta_n - \xi_{k+1,n}) \cdots (\zeta_n - \xi_{d,n})}{\xi_{r+1,n} \cdots \xi_{r+s,n}}. \end{aligned}$$

Assume  $\lim |\eta_n / \zeta_n| = \infty$ , and for  $i = t, \dots, r$ ,  $\lim |\eta_n / \xi_{i,n}| = 1$  so that  $\lim |\zeta_n / \xi_{i,n}| = 0$ . Because  $|\xi_{j,n}| > |\xi_{t,n}|$  for  $j < t$ , it follows that  $\lim |\zeta_n / \xi_{i,n}| = 0$  for all  $i \leq r$ . This implies that the product of the first  $r$  factors is  $(-1)^r$ . The product of the  $\alpha_i$ 's is finite by hypothesis so there are  $r+s$  finite factors. The finiteness of the limit of the left hand side therefore depends on the remaining factors some of which may tend to zero and others of which may tend to infinity. In particular, the terms between  $\alpha_{r+s}$  and the last fraction on the right all tend to infinity. The final fraction on the right has  $s$  terms in the denominator and  $d-k$  terms in the numerator. If  $d-k < s$  the fraction breaks into  $s$  factors each tending to a non-zero finite limit or infinity



and  $d - k - s$  factors that tend to infinity so that  $f_n(\zeta_n)$  cannot have finite limit contrary to hypothesis; thus  $s \leq d - k$ .

Now suppose  $\lim |\eta_n/\zeta_n| = \infty$  and consider

$$\frac{f_n(\eta_n)}{a_{d,n}\xi_{1,n} \cdots \xi_{k,n}} = \left(\frac{\eta_n}{\xi_{1,n}} - 1\right) \cdots \left(\frac{\eta_n}{\xi_{r,n}} - 1\right) \alpha_{r+1,n}(\eta_n) \cdots \alpha_{r+s,n}(\eta_n) \\ \left(\frac{\eta_n}{\xi_{r+s+1,n}} - 1\right) \cdots \left(\frac{\eta_n}{\xi_{k,n}} - 1\right) \frac{(\eta_n - \xi_{k+1,n}) \cdots (\eta_n - \xi_{d,n})}{\xi_{r+1,n} \cdots \xi_{r+d-k,n}} \frac{1}{\xi_{r+d-k+1} \cdots \xi_{r+s}}$$

where  $\alpha_j(\eta_n) = \eta_n - \xi_{j,n}$ .

Term by term, except for the last, the factors each grow at least as fast as the corresponding terms in the expression for  $f(\zeta_n)$ ; the last factor tends to zero at the same rate. Moreover, the  $\alpha_j(\eta_n)$  terms all tend to infinity so the product must tend to infinity as claimed.

If  $\lim |\eta_n/\zeta_n| = 0$ , all but the last factor grows slower than the corresponding factor and the  $\alpha_j(\eta_n)$  terms tend to zero so the product tends to zero as claimed.  $\square$

Now we can state our compactness result in full generality.

**Theorem 3.** *Given a post-singularly finite  $f \in \mathcal{T}E_{p,q}$  with bounded geometry the sequence  $E_n$  generated by Thurston iteration is contained in a compact subset of  $\mathcal{E}_{p,q}$ .*

In our discussion of functions in  $\mathcal{E}_{p,q}$  we had to normalize differently depending on whether or not the origin was a fixed point of the function. In the following theorem we make this distinction by writing the functions in  $\mathcal{E}_{p+\nu,q}$  as  $E(z) = \lambda z^\nu P(z) e^{Q(z)}$  with  $P(0) = 1$  so that the degree of the polynomial multiplying the exponential is  $p + \nu$  and  $\nu$  is the number of its zeros. Recall that if  $\nu = 0$ , the functions are normalized so that  $\lambda = 1$  and  $E(0) = 1$ . If  $\nu > 0$ , the functions are normalized so that for a chosen critical point  $\xi$  with non-zero critical value,  $\lambda$  is defined by the condition  $E(\xi) = 1$ .

*Proof.* The special case  $P_n \equiv 1$  is the content of theorem 2. We begin here with the proof of the theorem for  $\nu = 0$  and  $p > 0$ :

$$E_n(z) = (a_{p,n}z^p + a_{p-1,n}z^{p-1} + \cdots + 1)e^{Q_n(z)}.$$

We have  $E'_n = f_n e^{Q_n}$  where

$$f_n(z) = P_n(z)Q'_n(z) + P'_n(z) = c_{q+p-1,n}z^{q+p-1} + \cdots + c_{0,n}$$

is the critical polynomial.

The coefficients  $c_{m,n}$  are

$$(12) \quad c_{0,n} = a_{1,n} + b_{1,n}, \quad c_{1,n} = 2b_{2,n} + a_{1,n}b_{1,n} + 2a_{2,n},$$

$$(13) \quad c_{m,n} = \left( \sum_{k=1}^m kb_{k,n}a_{m-k,n} \right) + ma_{m,n}, \quad m = 2 \dots, q+p-2,$$

$$(14) \quad c_{q+p-1,n} = qa_{p,n}b_{q,n}.$$

with the convention that for  $m > p$ ,  $a_{m,n} = 0$  and for  $m > q$ ,  $b_{m,n} = 0$ .

We first rule out the case that  $\lim_{n \rightarrow \infty} \max\{|a_{i,n}|\} = 0$  and  $\lim_{n \rightarrow \infty} \max\{|b_{m,n}|\} = 0$ . On any compact set,  $P_n$  and  $Q_n$  converge uniformly to 0 so that  $\lim E_n(1) = 0$  violating bounded geometry.

Assume that all the  $a_{i,n}$  and  $b_{m,n}$  have finite limits. If neither the limit of  $a_{p,n}$  nor the limit of  $b_{q,n}$  is zero, the functions  $E_n$  converge to a function in the space.

Recall that if  $r$  is a root of  $P_n$  but not a root of  $P'_n$  it cannot be a root of  $f_n$  so that any root of  $P_n$  that is a root of  $f_n$  must be a double root of  $P_n$ . As we did for  $Q_n$ , taking out factors with multiple roots, we write  $P_n = D_n S_n$  and  $P'_n = D_n R_n$  so that  $D_n$  is monic, both  $D_n$  and  $S_n$  have factors with the multiple roots but the roots of  $R_n$  are not roots of  $P_n$ . Then

$$f_n = D_n(S_n Q'_n + R_n)$$

and if the degree of  $f_n$  goes down, the degree of  $S_n Q'_n + R_n$  goes down too. It follows from lemma 6 that if the degree of  $f_n$  goes down there is a sequence  $\xi_n$  of roots of  $S_n Q'_n + R_n$  that converge to infinity; by definition, these are not roots of  $P_n$ . Notice that since the degree of  $f_n$  is  $p+q-1$ , and 0 is a root of  $Q_n$  but not a root of  $P_n$ ,  $P_n$  and  $Q_n$  can have at most  $\max\{p-1, q-1\} < p+q-1$  common roots.

In order for the degree of  $f_n$  to go down,  $\lim c_{q+p-1,n} = 0$  so either (1)  $a_p = \lim a_{p,n} \neq 0$  and  $b_q = \lim b_{q,n} = 0$  or (2)  $a_p = \lim a_{p,n} = 0$  and  $b_q = \lim b_{q,n} \neq 0$  or (3) both limits are 0. In any case, there is a sequence of critical points  $\xi_n$  converging to infinity that are not roots of  $P_n$ . In (1),  $\lim P_n$  is a finite polynomial of degree  $p$  and  $\lim P_n(\xi_n)$  converges to infinity. In (2),  $\lim Q_n$  is a finite polynomial of degree  $q$ , and the roots of  $Q_n$  converge to finite values. Since the sequence of critical points  $\xi_n$  converges to infinity, for large  $n$ ,  $\xi_n$  is not a root of  $Q_n$ . We thus conclude  $Q_n(\xi_n)$  converges to infinity.

In (1), whether  $\lim Q_n(\xi_n)$  is bounded or  $\lim Q_n(\xi_n)$  is unbounded in a Julia direction so that  $e^{Q_n}$  remains bounded,  $\lim E_n(\xi_n) = \infty$ ; if  $\lim Q_n(\xi_n)$  is unbounded in a non-Julia direction  $e^{Q_n}$  has limit 0 or  $\infty$ . Since  $e^{Q_n(\xi_n)}$  is the dominant term in  $E_n(\xi_n)$ ,  $\lim E_n(\xi_n)$  is 0 or  $\infty$ . Bounded geometry rules out each of these possibilities so the first case cannot occur.

In (2), if  $\lim Q_n(\xi_n)$  is unbounded in a non-Julia direction  $E_n(\xi_n)$  goes to 0 or  $\infty$ . If  $\lim Q_n(\xi_n)$  is unbounded in a non-Julia direction and  $P_n(\xi_n)$  goes to 0 or  $\infty$  then  $E_n(\xi_n)$  does the same. Bounded geometry rules out these possibilities. If  $\lim Q_n(\xi_n)$  is unbounded in a non-Julia direction and  $P_n(\xi_n)$  is finite then  $E_n(\xi_n)$  remains finite

but violates the topological constraints; that is, at least one of the winding numbers in  $\eta_n$  goes to  $\infty$ .

In (3), since both limits  $a_p$  and  $b_q$  are zero,  $qa_{p,n}b_{q,n}$  goes to zero faster than either  $a_{p,n}$  or  $qb_{q,n}$  separately. Let  $\xi_{1,n}, \dots, \xi_{k,n}$  be all the roots of  $S_n Q'_n + R_n$  that converge to infinity. If, for any  $j = 1, \dots, k$ ,  $\lim P_n(\xi_{j,n}) = \infty$  then, arguing as above,  $\lim E_n(\xi_{j,n})$  is either 0 or  $\infty$ ; if  $\lim P_n(\xi_{j,n}) = 0$ , again by the above,  $\lim E_n(\xi_{j,n})$  is either 0 or  $\infty$ ; and arguing yet again as above, if  $\lim Q_n(\xi_{j,n}) = \infty$  either  $E_n(\xi_{j,n})$  converges to 0 or  $\infty$  or the topological constraint is violated,  $\eta_n \rightarrow \infty$ .

We claim that these are the only possible cases. Suppose not; that is, suppose that for all  $j = 1, \dots, k$ ,  $\lim P_n(\xi_{j,n})$  is finite and non-zero and  $\lim Q_n(\xi_{j,n})$  is finite. This implies that to each  $j$  there must be a root  $r_{j,n}$  of  $P_n$  converging to infinity such that  $\lim(\xi_{j,n} - r_{j,n}) = \alpha_j$ , where  $\alpha_j$  is finite and non-zero. By proposition 3, if the roots converge at different rates, for one of the roots we would have  $\lim P_n(\xi_{j,n}) = 0$  or  $\infty$ .

Now we can apply proposition 2 to  $P_n$ , to conclude that  $k = p/2$ , and that the growth rate of each  $r_{j,n}$  is  $1/k$ ; that is, there exists a positive constant  $C_1$  such that

$$C_1^{-1}|a_{p,n}^{-1/k}| \leq \left| \frac{1}{r_{j,n}} \right| \leq C_1|a_{p,n}^{-1/k}|$$

and therefore

$$C_1^{-1}|a_{p,n}^{-1/k}| \leq \left| \frac{1}{\xi_{j,n}} \right| \leq C_1|a_{p,n}^{-1/k}|.$$

This implies

$$C_1^{-k}|a_{p,n}^{-1}| \leq \Pi_1^k \left| \frac{1}{\xi_{j,n}} \right| \leq C_1^k|a_{p,n}^{-1}|.$$

Applying the proposition to the critical polynomial we have

$$C_2^{-1}|(qa_{p,n}b_{q,n})|^{-1} \leq \left| \Pi_1^k \frac{1}{\xi_{j,n}} \right| \leq C_2|(qa_{p,n}b_{q,n})|^{-1}$$

Together these imply  $\lim b_{q,n} \neq 0$ , contradicting our assumption.

Since either bounded geometry or the topological constraint is violated, the coefficients of the critical polynomial  $f_n$  cannot remain bounded if the degree of the polynomial goes down.

The final step is to rule out the possibility that some of the coefficients of  $f_n$  tend to infinity. If this happens, either some  $a_{i,n}$ ,  $i = 1, \dots, p$  or some  $b_{m,n}$ ,  $m = 1, \dots, q$  tend to  $\infty$ . Again there are several cases. (1) 1 is not a root of  $P_n$  and some  $a_{i,n}$  goes to infinity. In this case,  $P_n(1)$  does too and bounded geometry is violated. (2) All the  $a_{i,n}$  remain finite and 1 is not a root of  $Q_n$ . In this case,  $Q_n(1)$  goes to infinity and bounded geometry or the topological constraint is violated. (3) The remaining possibilities: 1 is a root of  $P_n$  or if it is not a root of  $P_n$  and all the  $a_{i,n}$  remain finite but 1 is a root of  $Q_n$ . To deal with this case, we look at critical points.

Let  $\xi_{j,n}$  be a sequence of critical points that are not roots of  $P_n$ . If, for some  $\xi_{i,n}$ ,  $P_n(\xi_{i,n})$  converges to either zero or infinity then again bounded geometry is

violated. Similarly, if  $Q_n(\xi_{i,n})$  converges to infinity, either bounded geometry or the topological constraint is violated.

If any of the  $\xi_{i,n}$  remain finite then either  $P_n(\xi_{i,n})$  or  $Q_n(\xi_{i,n})$  converges to infinity so we may assume that all the  $\xi_{i,n}$  converge to infinity. If, for some  $i, j$ ,  $\lim |\xi_{i,n}/\xi_{j,n}|$  converges to zero or infinity, then one of  $P_n(\xi_{i,n})$  or  $P_n(\xi_{j,n})$  converges to zero or infinity. So we may assume the critical points all converge to infinity at the same rate and since we are assuming  $P_n$  and  $Q_n$  converge to finite values for these points, each must be asymptotic to both a root of  $P_n$  and a root of  $Q_n$  and so these roots are not finite.

Suppose  $B_n = \max\{|a_{i,n}|, |b_{j,n}|, |c_{k,n}|\}$  and write  $P_n = B_n g_{1,n}$ ,  $Q_n = B_n g_{2,n}$  and  $f_n = B_n g_{3,n}$ . Now since the  $g_{k,n}$  have bounded coefficients we can apply proposition 3 to  $g_{1,n}$  to deduce that each  $\xi_{i,n}$  must grow like  $|a_{p,n}/B_n|^{2/p}$ . Applying it to  $g_{2,n}$  we deduce that each  $\xi_{i,n}$  grows like  $|b_{q,n}/B_n|^{2/q}$  and applying it to  $g_{3,n}$  we deduce that each  $\xi_{i,n}$  grows like  $|qa_{p,n}b_{q,n}/B_n|^{2/(p+q-1)}$ . This gives us the contradiction for this case so either bounded geometry fails or the topological constraint is violated when the coefficients in  $E_n$  are unbounded and the theorem is proved for  $\nu = 0$ .

We turn now to the situation where  $\nu > 0$  and the origin is a fixed point of  $E_n$ . Recall our normalization:  $E_n(z) = \lambda_n z^\nu P_n(z) e^{Q_n(z)}$ ,  $\nu \geq 1$ , is a sequence of entire functions with  $P_n$  and  $Q_n$  polynomials of degrees  $p$  and  $q$  respectively such that  $P(0) = 1$ . We choose a critical point  $\xi$  of  $f$  such that  $f(\xi) \neq 0$ . Then  $\xi_n = w^{\mu_n}(\xi)$  is a critical point of  $E_n$  such that  $E_n(\xi_n) \neq 0$  (and hence  $P_n(\xi_n) \neq 0$ ) and we normalize so that  $\lambda_n$  is defined by the condition  $E_n(\xi_n) = 1$ .

We first treat the case  $\nu > 0$ ,  $p = 0$  so that  $P_n \equiv 1$ .

Set  $F_n = z^\nu e^Q$  so that  $E_n = \lambda_n F_n$ . Both  $E_n$  and  $F_n$  have same critical points; zero is a critical point of multiplicity  $\nu - 1$  and the other critical points are roots of the critical polynomial  $f_n = (zQ'_n + \nu)$ . It has degree  $q$  and the coefficient  $c_{m,n}$  of  $z^m$  is

$$c_{m,n} = mb_{m,n}, m = 1, \dots, q, \quad c_{0,n} = \nu$$

Since  $Q_n$  also has  $q$  roots and one of them is 0, there is at least one root  $\xi_n$  of  $f_n$  that is not a root of  $Q_n$ . We choose such a sequence  $\xi_n$  of roots of  $f_n$  and set  $\lambda_n = (\xi_n^\nu e^{Q_n(\xi_n)})^{-1}$ .

Since  $P_n \equiv 1$  we need only consider the coefficients  $b_{m,n}$  of  $Q_n$ . If  $\lim \max\{|b_{m,n}|\}$  is 0, then on any compact set both  $Q_n$  and  $Q'_n$  converge uniformly to 0. Therefore  $f_n$  converges to a constant and the critical points converge to zero or infinity. Thus, for the chosen non-zero critical point,  $\xi_n$ , either  $\xi_n \rightarrow 0$  and  $\lambda_n \rightarrow \infty$  or  $\xi_n \rightarrow \infty$  and  $\lambda_n \rightarrow 0$ . This implies  $E_n(1) = \lambda_n e^{Q_n(1)} \rightarrow 0$  or  $\infty$  violating bounded geometry.

Suppose all the  $b_{m,n}$  remain bounded in the limit but  $\lim_{n \rightarrow \infty} \max\{|b_{m,n}|\} \neq 0$ . If  $b_q = \lim b_{q,n} \neq 0$ ,  $E_n$  converges to a function in the space.

If  $b_q = 0$ , there is at least one sequence of roots of  $f_n$ ,  $\xi_n \rightarrow \infty$  such that  $\xi_n$  is not a root of  $Q_n$ . We choose one such sequence for our normalization. Because the exponential dominates, we have  $F_n(\xi_n) = \xi_n^\nu e^{Q_n(\xi_n)} \rightarrow 0$  or  $\infty$ .

Now

$$\lambda_n = \xi_n^{-\nu} e^{-Q_n(\xi_n)} \rightarrow \infty \text{ or } 0$$

so that

$$E_n(1) = \xi_n^{-\nu} e^{-Q_n(\xi_n) + Q_n(1)} \rightarrow \infty \text{ or } 0.$$

We see that bounded geometry is violated proving the theorem for  $P_n \equiv 1$ .

The last case is  $\nu > 0$  and  $p > 0$ . The critical polynomial is  $g_n = z^{\nu-1} f_n$  where

$$f_n(z) = \nu P_n(z) + z(P'_n(z) + Q'_n(z)P_n(z)).$$

The non-zero critical points of  $F_n$  are roots of  $f_n$ . The degree of  $f_n$  is  $p + q$ .

The coefficient of  $e_{m,n}$  of  $z^m$  in  $f_n$  is

$$e_{m,n} = c_{m-1,n} + \nu a_{m,n}$$

where  $c_{m,n}$  are the coefficients in equations (12) and again our convention is that  $b_{m,n} = 0$  for  $m > q$  and  $a_{m,n} = 0$  for  $m > p$ .

As we did for the  $\nu = 0$  case, we divide the argument into cases depending on what happens to the coefficients of the  $P_n$  and  $Q_n$ . We first rule out the case that  $\lim_{n \rightarrow \infty} \max\{|a_{i,n}|\} = 0$  and  $\lim_{n \rightarrow \infty} \max\{|b_{m,n}|\} = 0$ . In this case, on any compact set,  $P_n$ ,  $Q_n$  and  $Q'_n$  converge uniformly to 0. Any choice of a non-zero critical point  $\xi_n$  converges to 0 or  $\infty$ . Therefore  $\lim F_n(1) \rightarrow 0$ . Because the exponential term dominates,  $\lambda_n = F_n(\xi_n)^{-1} = \xi_n^{-\nu} (P_n(\xi_n) e^{Q_n(\xi_n)})^{-1} \rightarrow 0$  or  $\infty$ , and  $E_n(1) \rightarrow 0$  or  $\infty$  violating bounded geometry and ruling out this case.

Now assume that all the  $a_{i,n}$  and  $b_{j,n}$  have finite limits (so that the  $c_{m,n}, e_{m,n}$  do also) but that either  $\lim_{n \rightarrow \infty} \max\{|a_{i,n}|\} \neq 0$  or  $\lim_{n \rightarrow \infty} \max\{|b_{m,n}|\} \neq 0$ . If the degree of  $f_n$  decreases, we must have  $e_{p+q,n} \rightarrow 0$  and either  $a_p = \lim a_{p,n} = 0$  or  $b_q = \lim b_{q,n} = 0$ . As in the  $\nu = 0$  case, we can find a sequence  $\xi_n$  of roots of  $f_n$  that are not roots of  $P_n$  which converge to infinity. We choose such a sequence for the normalization. Applying the arguments we used above when  $\nu = 0$ , we see that either  $P_n(\xi_n) e^{Q_n(\xi_n)} \rightarrow 0$  or  $\infty$ , or it remains bounded, but the topological constraint is violated. We compute  $\lambda_n$  for each of these possibilities and see that

$$\lambda_n = \xi_n^{-\nu} P_n(\xi_n)^{-1} e^{-Q_n(\xi_n)} \rightarrow \infty \text{ or } 0.$$

Therefore if 1 is not a root of  $P_n$  we have  $E_n(1) = \lambda_n F_n(1) \rightarrow 0$  or  $\infty$ . If 1 is a root of  $P_n$ , the critical point  $\xi_n$  has the orbit  $\xi_n, 1, 0$ . If there are no other critical points, the post-singular set has only three points and we have excluded this trivial case. Thus there must be another critical point  $\tilde{\xi}_n$  whose critical value  $\tilde{v}_n$  is neither 0 nor 1. We have

$$\tilde{v}_n = E_n(\tilde{\xi}_n) = \lambda_n \tilde{\xi}_n^{-\nu} P_n(\tilde{\xi}_n) e^{Q_n(\tilde{\xi}_n)}.$$

If  $\tilde{\xi}_n$  remains finite,  $\tilde{v}_n$  converges to 0 or  $\infty$  violating bounded geometry. If  $\xi_n$  and  $\tilde{\xi}_n$  converge to  $\infty$  at different rates, because  $\lambda_n$  depends on the growth rate of  $\xi_n$ ,  $\tilde{v}_n$  converges either to 0 or  $\infty$ . If they converge at the same rate,  $\tilde{v}_n$  may converge to a

finite value  $\tilde{v}$ . Now both  $P_n(\tilde{v}_n)$  and  $e^{Q_n(\tilde{v}_n)}$  remain bounded since their coefficients remain finite. This implies either  $\tilde{v}_n \rightarrow 0$  or  $1$ , or, since  $\lambda_n \rightarrow 0$  or  $\infty$ ,

$$E_n(\tilde{v}_n) = \lambda_n \tilde{v}_n^\nu P_n(\tilde{v}_n) e^{Q_n(\tilde{v}_n)} \rightarrow \infty \text{ or } 0$$

violating bounded geometry. It follows that the coefficients of  $P_n$  and  $Q_n$  can only remain bounded if the degree does not go down.

If either the coefficients of  $P_n$  or  $Q_n$  are unbounded in the limit,  $\lambda_n \rightarrow \infty$  or  $0$ . We saw in the  $\nu = 0$  case that either  $P_n(\xi_n) e^{Q_n(\xi_n)} \rightarrow 0$  or  $\infty$ , or it remains bounded but the topological constraint is violated. Again

$$\lambda_n = \xi_n^{-\nu} P_n(\xi_n)^{-1} e^{-Q_n(\xi_n)} \rightarrow \infty \text{ or } 0$$

so that  $E_n(1) \rightarrow \infty$  or  $0$  unless  $1$  is a root of  $P_n$ . If  $1$  is a root, we again let  $\tilde{\xi}_n$  be a different critical point whose critical value  $\tilde{v}_n$  is neither  $0$  nor  $1$ . If  $\tilde{v}_n$  converges to  $0, 1$  or  $\infty$ , bounded geometry is violated. If not, we consider  $E_n(\tilde{v}_n)$ . Now  $\tilde{v}_n$  remains bounded and  $\xi_n$  converges to  $\infty$ . If  $\lim_{n \rightarrow \infty} \max\{|a_{i,n}|\} = \infty$ , both  $P_n(\xi_n)$  and  $P_n(\tilde{v}_n)$  converge to  $\infty$  but at different rates:  $\lim P_n(\tilde{v}_n)/P_n(\xi_n) = 0$  and similarly if  $\lim_{n \rightarrow \infty} \max\{|b_{m,n}|\} = \infty$ ,  $\lim Q_n(\xi_n)/Q_n(\tilde{v}_n) = \infty$ . It follows that

$$E_n(\tilde{v}_n) = \frac{\tilde{v}_n^\nu P_n(\tilde{v}_n)}{\xi_n^\nu P_n(\xi_n)} e^{Q_n(\tilde{v}_n) - Q_n(\xi_n)} \rightarrow \infty \text{ or } 0$$

and again bounded geometry is violated ruling out this possibility and completing the proof of the theorem. □

## 10. THE MAIN RESULT

The main result in this paper is that

**Theorem 4** (Main Theorem). *A post-singularly finite topological exponential map  $f \in \mathcal{TE}_{p,q}$  is combinatorially equivalent to a unique  $(p, q)$ -exponential map  $E = Pe^Q$  if and only if  $f$  has bounded geometry.*

*Necessity.* If  $f$  is combinatorially equivalent to  $E = Pe^Q$ , then  $\sigma$  has a unique fixed point  $\tau_0$  so that  $\tau_n = \tau_0$  for all  $n$ . The complex structure on  $\hat{\mathbb{C}} \setminus P_f$  defined by  $\tau_0$  induces a hyperbolic metric on it. The shortest geodesic in this metric gives a lower bound on the lengths of all geodesics so that  $f$  satisfies the hyperbolic definition of bounded geometry. □

The proof of sufficiency is more complicated and needs some preparatory material. We defer it to the next section.

**10.1. Proof of the Main Result - Sufficiency.** Suppose  $f$  is a post-singularly finite topological exponential map in  $\mathcal{TE}_{p,q}$ . For any  $\tau = [\mu] \in T_f$ , let  $T_\tau T_f$  and  $T_\tau^* T_f$  be the tangent space and the cotangent space of  $T_f$  at  $\tau$  respectively. Let  $w^\mu$  be the corresponding normalized quasiconformal map fixing  $0, 1, \infty$ . Then  $T_\tau^* T_f$  coincides with the space  $\mathcal{Q}_\mu$  of integrable meromorphic quadratic differentials  $q = \phi(z)dz^2$ . Integrability means that the norm of  $q$ , defined by

$$\|q\| = \int_{\hat{\mathbb{C}}} |\phi(z)| dz d\bar{z}$$

is finite. This condition implies that the poles of  $q$  must occur at points of  $w^\mu(P_f)$  and that these poles are simple.

Set  $\tilde{\tau} = \sigma(\tau) = [\tilde{\mu}]$  and denote by  $w^\mu$  and  $w^{\tilde{\mu}}$  the corresponding normalized quasiconformal maps. We have the following commutative diagram:

$$\begin{array}{ccc} \hat{\mathbb{C}} \setminus f^{-1}(P_f) & \xrightarrow{w^{\tilde{\mu}}} & \hat{\mathbb{C}} \setminus w^{\tilde{\mu}}(f^{-1}(P_f)) \\ \downarrow f & & \downarrow E_{\mu, \tilde{\mu}} \\ \hat{\mathbb{C}} \setminus P_f & \xrightarrow{w^\mu} & \hat{\mathbb{C}} \setminus w^\mu(P_f). \end{array}$$

Note that in the diagram, by abuse of notation, we write  $f^{-1}(P_f)$  for  $f^{-1}(P_f \setminus \{\infty\}) \cup \{\infty\}$ . Since by definition  $\tilde{\mu} = f^* \mu$ , the map  $E = E_{\mu, \tilde{\mu}} = w^\mu \circ f \circ (w^{\tilde{\mu}})^{-1}$  defined on  $\hat{\mathbb{C}}$  is analytic. By Theorem 1,  $E_{\mu, \tilde{\mu}} = P_{\tau, \tilde{\tau}} e^{Q_{\tau, \tilde{\tau}}}$  for a pair of polynomials  $P = P_{\tau, \tilde{\tau}}$  and  $Q = Q_{\tau, \tilde{\tau}}$  of respective degrees  $p$  and  $q$ .

Let  $\sigma_* : T_\tau T_f \rightarrow T_{\tilde{\tau}} T_f$  and  $\sigma^* : T_\tau^* T_f \rightarrow T_{\tilde{\tau}}^* T_f$  be the tangent and co-tangent map of  $\sigma$ , respectively.

Let  $\beta(t) = [\mu(t)]$  be a smooth path in  $T_f$  passing through  $\tau$  at  $t = 0$  and let  $\eta = \beta'(0)$  be the corresponding tangent vector at  $\tau$ . Then the pull-back  $\tilde{\beta}(t) = [f^* \mu(t)]$  is a smooth path in  $T_f$  passing through  $\tilde{\tau}$  at  $t = 0$  and  $\tilde{\eta} = \sigma_* \eta = \tilde{\beta}'(t)$  is the corresponding tangent vector at  $\tilde{\tau}$ . We move these tangent vectors to the origin in  $T_f$  using the maps

$$\eta = (w^\mu)^* \nu \quad \text{and} \quad \tilde{\eta} = (w^{\tilde{\mu}})^* \tilde{\nu}.$$

This gives us the following commutative diagram:

$$\begin{array}{ccc} \tilde{\eta} & \xleftarrow{(w^{\tilde{\mu}})^*} & \tilde{\xi} \\ \uparrow f^* & & \uparrow E^* \\ \eta & \xleftarrow{(w^\mu)^*} & \xi \end{array}$$

Now suppose  $\tilde{q}$  is a co-tangent vector in  $T_{\tilde{\tau}}^*$  and let  $q = \sigma^* \tilde{q}$  be the corresponding co-tangent vector in  $T_\tau^*$ . Then  $\tilde{q} = \tilde{\phi}(w)dw^2$  is an integrable quadratic differential on  $\hat{\mathbb{C}}$  that can have at worst simple poles along  $w^{\tilde{\mu}}(P_f)$  and  $q = \phi(z)dz^2$  is an integrable quadratic differential on  $\hat{\mathbb{C}}$  that can have at worst simple poles along  $w^\mu(P_f)$ . This implies that  $q = \sigma_* \tilde{q}$  is also the push-forward integrable quadratic differential

$$q = E_* \tilde{q} = \phi(z)dz^2$$

of  $\tilde{q}$  by  $E$ . To see this, recall from section 3 that  $E$ , and a choice of curves  $L_i$  from the branch points, determine a finite set of domains  $W_i$  on which  $E$  is an unbranched covering to a domain homeomorphic to  $\mathbb{C}^*$ . Since  $E$  restricted to each  $W_i$  is either a topological model for  $e^z$  or  $z^k$ , we may divide each  $W_i$  into a collection of fundamental domains on which  $E$  is bijective. Therefore the coefficient  $\phi(z)$  of  $q$  is given by the formula

$$(15) \quad \phi(z) = (\mathcal{L}\tilde{\phi})(z) = \sum_{E(w)=z} \frac{\tilde{\phi}(w)}{(E'(w))^2} = \frac{1}{z^2} \sum_{E(w)=z} \frac{\tilde{\phi}(w)}{(\frac{P'(w)}{P(w)} + Q'(w))^2}$$

where  $\mathcal{L}$  is the standard transfer operator and  $\tilde{\phi}$  is the coefficient of  $\tilde{q}$ . Thus

$$(16) \quad q = \phi(z)dz^2 = \frac{dz^2}{z^2} \sum_{E(w)=z} \frac{\tilde{\phi}(w)}{(\frac{P'(w)}{P(w)} + Q'(w))^2}$$

It is clear that as a quadratic differential defined on  $\hat{\mathbb{C}}$ , we have

$$\|q\| \leq \|\tilde{q}\|.$$

Since  $q$  is integrable and 0 and  $\infty$  are isolated singularities, it follows that  $q$  has at worst possible simple poles at these points so that the inequality holds on all of  $\hat{\mathbb{C}}$ .

By formula (15), we have

$$\langle \tilde{q}, \tilde{\xi} \rangle = \langle q, \xi \rangle$$

which implies

$$\|\tilde{\eta}\| \leq \|\eta\|$$

where this is the  $L^\infty$  norm. This gives another proof of Lemma 2. Furthermore, we have the following stronger assertion

**Lemma 8.**

$$\|q\| < \|\tilde{q}\|$$

and

$$\|\tilde{\eta}\| < \|\eta\|.$$

*Proof.* Suppose there is a  $\tilde{q}$  such that  $\|q\| = \|\tilde{q}\| \neq 0$ . Using the change of variable  $E(w) = z$  on each fundamental domain we get

$$\begin{aligned} \int_{\hat{\mathbb{C}}} \left| \sum_{E(w)=z} \frac{\tilde{\phi}(w)}{(E'(w))^2} \right| dz d\bar{z} &= \int_{\hat{\mathbb{C}}} |\phi(z)| dz d\bar{z} = \int_{\hat{\mathbb{C}}} |\tilde{\phi}(w)| dw d\bar{w} \\ &= \sum_i \int_{W_i} |\tilde{\phi}(w)| dw d\bar{w} = \int_{\hat{\mathbb{C}}} \sum_i \left| \frac{\tilde{\phi}(w)}{(E'(w))^2} \right| dz d\bar{z}. \end{aligned}$$

By the triangle inequality, all the factors  $\frac{\tilde{\phi}(w)}{(E'(w))^2}$  in  $\sum_{E(w)=z} \frac{\tilde{\phi}(w)}{(E'(w))^2}$  have the same argument. That is, there is a real number  $a_z$  for every  $z$  such that for any pair of



points  $w, w'$  with  $E(w) = E(w') = z$ ,

$$\frac{\tilde{\phi}(w)}{(E'(w))^2} = a_z \frac{\tilde{\phi}(w')}{(E'(w'))^2}.$$

Now formula (15) implies  $\phi(z) = \infty$  which cannot be which proves the lemma.  $\square$

**Remark 2.** *The real point here is that  $E$  has infinite degree and any  $q$  has finitely many poles. If there were a  $\tilde{q}$  with  $\|q\| = \|\tilde{q}\| \neq 0$  and if  $Z$  is the set of poles of  $\tilde{q}$ , then the poles of  $q$  would be contained in the set  $E(Z) \cup \mathcal{V}_E$ , where  $\mathcal{V}_E$  is the set of critical values of  $E$ . Thus, by formula (15),*

$$E^*q = \phi(E(w))dw^2 = d\tilde{q}(w),$$

where  $d$  is the degree of  $E$ . Furthermore,

$$E^{-1}(E(Z) \cup \mathcal{V}_E) \subseteq Z \cup \Omega_E.$$

Since  $d$  is infinite, the last inclusion formula can not hold since the left hand side is infinite and the right hand side is finite.

An immediate corollary is

**Corollary 1.** *For any two points  $\tau$  and  $\tilde{\tau}$  in  $T_f$ ,*

$$d_T(\sigma(\tau), \sigma(\tilde{\tau})) < d_T(\tau, \tilde{\tau}).$$

Furthermore,

**Lemma 9.** *If  $\sigma$  has a fixed point in  $T_f$ , then this fixed point must be unique. This is equivalent to saying that a post-singularly finite  $f$  in  $\mathcal{T}E_{p,q}$  is combinatorially equivalent to at most one  $(p, q)$ -exponential map  $E = Pe^Q$ .*

We can now finish the proof of the main theorem.

*Proof of Sufficiency.* Suppose that  $f$  has bounded geometry. Recall that the map defined by

$$(17) \quad E_n = w^{\mu_n} \circ f \circ (w^{\mu_{n+1}})^{-1}$$

is a  $(p, q)$ -exponential map.

If  $q = 0$ ,  $E_n$  is a polynomial and the theorem follows from the arguments given in [CJ1] and [DH]. Note that if  $P_f = \{0, 1, \infty\}$ , then  $f$  is a universal covering map of  $\mathbb{C}^*$  and is therefore combinatorially equivalent to  $e^{z-1}$ . Thus in the following argument, we assume that  $\#(P_f) \geq 4$ . Then, given our normalization conventions and the bounded geometry hypothesis we see that the functions  $E_n$ ,  $n = 0, 1, \dots$  satisfy the following conditions:

- 1)  $m = \#(w^{\mu_n}(P_f)) \geq 4$  is fixed.
- 2)  $0, 1, \infty \in w^{\mu_n}(P_f)$ .
- 3)  $\Omega_{E_n} \cup \{0, 1, \infty\} \subseteq E_n^{-1}(w^{\mu_n}(P_f))$ .
- 4) there is a  $b > 0$  such that  $d_{sp}(p_n, q_n) \geq b$  for any  $p_n, q_n \in w^{\mu_n}(P_f)$ .

As a consequence of theorem 3 we have

**Corollary 2.** *Suppose  $f \in \mathcal{T}E_{p,q}$  has bounded geometry and  $\mathbb{E}$  is the corresponding family  $E_n \in \mathcal{E}_{p,q}$  defined by equation (17). Then  $\mathbb{E}$  is normal. That is, there is a subsequence  $E_{n_i}$  converging to a map  $E = Pe^Q \in \mathcal{E}_{p,q}$  where  $P$  and  $Q$  are polynomials of degrees  $p$  and  $q$  respectively.*

Any integrable quadratic differential  $q_n \in T_{\tau_n}^* \mathcal{T}_f$  has, at worst, simple poles in the finite set  $P_{n+1,f} = w^{\mu_{n+1}}(P_f)$ . Since  $T_{\tau_n}^* \mathcal{T}_f$  is a finite dimensional linear space, there is a quadratic differential  $q_{n,max} \in T_{\tau_n}^* \mathcal{T}_f$  with  $\|q_{n,max}\| = 1$  such that

$$0 \leq a_n = \sup_{\|q_n\|=1} \|(E_n)_* q_n\| = \|(E_n)_* q_{n,max}\| < 1.$$

Moreover, by the bounded geometry condition, the possible simple poles of  $\{q_{n,max}\}_{n=1}^\infty$  lie in a compact set and hence these quadratic differentials lie in a compact subset of the space of quadratic differentials on  $\hat{\mathbb{C}}$  with, at worst, simple poles at  $m = \#(P_f)$  points.

Let

$$a_{\tau_0} = \sup_{n \geq 0} a_n.$$

Let  $\{n_i\}$  be a sequence of integers such that the subsequence  $a_{n_i} \rightarrow a_{\tau_0}$  as  $i \rightarrow \infty$ . By compactness,  $\{E_{n_i}\}_{i=0}^\infty$  has a convergent subsequence, (for which we use the same notation) that converges to a holomorphic map  $E \in \mathcal{E}_{p,q}$ . Taking a further subsequence if necessary, we obtain a convergent sequence of sets  $P_{n_i,\tau_0} = w^{\mu_{n_i}}(P_f)$  with limit set  $X$ . By bounded geometry,  $\#(X) = \#(P_f)$  and  $d_{sp}(x, y) \geq b$  for any  $x, y \in X$ . Thus we can find a subsequence  $\{q_{n_i,max}\}$  converging to an integrable quadratic differential  $q$  of norm 1 whose only poles lie in  $X$  and are simple. Now by lemma 8,

$$a_{\tau_0} = \|E_* q\| < 1.$$

Thus we have proved that there is an  $0 < a_{\tau_0} < 1$ , depending only on  $b$  and  $f$ , such that

$$\|\sigma_*\| \leq \|\sigma^*\| \leq a_{\tau_0}.$$

Let  $l_0$  be a curve connecting  $\tau_0$  and  $\tau_1$  in  $T_f$  and set  $l_n = \sigma_f(l_0)$  for  $n \geq 1$ . Then  $l = \cup_{n=0}^\infty l_n$  is a curve in  $T_f$  connecting all the points  $\{\tau_n\}_{n=0}^\infty$ . For each point  $\tilde{\tau}_0 \in l_0$ , we have  $a_{\tilde{\tau}_0} < 1$ . Taking the maximum gives a uniform  $a < 1$  for all points in  $l_0$ . Since  $\sigma$  is holomorphic,  $a$  is an upper bound for all points in  $l$ . Therefore,

$$d_T(\tau_{n+1}, \tau_n) \leq a d_T(\tau_n, \tau_{n-1})$$

for all  $n \geq 1$ . Hence,  $\{\tau_n\}_{n=0}^\infty$  is a convergent sequence with a unique limit point  $\tau_\infty$  in  $T_f$  and  $\tau_\infty$  is a fixed point of  $\sigma$ .  $\square$

## 11. CONCLUDING REMARKS

One can formally define a Thurston obstruction for a post-singularly finite  $(p, q)$ -topological exponential map  $f$  with  $q \geq 1$ . Because such an  $f$  is a branched covering of infinite degree, however, many arguments in the proof of the Thurston theorem in [DH] and [CJ1] that use the finiteness of the covering in an essential way, do not apply. Thus, how to use Thurston obstruction to characterize a  $(p, q)$ -topological exponential map  $f$  is not very clear to us. We can, however, define an analog of the *canonical* Thurston obstruction for a  $(p, q)$ -topological exponential map  $f$  which depends on the hyperbolic lengths of curves.

Let  $\sigma$  be the induced map on the Teichmüller space  $T_f$ . For any  $\tau_0 \in T_f$ , and for  $n \geq 1$ , let  $\tau_n = \sigma^n(\tau_0)$ . Let  $\gamma$  denote a simple closed non-peripheral curve in  $\mathbb{C} \setminus P_f$ . Define

$$\Gamma_c = \{\gamma \mid \forall \tau_0 \in T_f, l_{\tau_n}(\gamma) \rightarrow 0 \text{ as } n \rightarrow \infty\}.$$

We have that

**Corollary 3.** *If  $\Gamma_c \neq \emptyset$ , then  $f$  has no bounded geometry and therefore,  $f$  is not combinatorially equivalent to a  $(p, q)$ -exponential map.*

The converse should be also true but we have no proof at this time. The reason is that in the characterization of the post-critically finite case for rational maps, many arguments in the proof of this converse use the finiteness of the covering in an essential way (see [Pi, CJ2]).

A *Levy cycle* is a special Thurston obstruction for rational maps. It can be defined for a  $(p, q)$ -topological exponential map  $f$  as follows. A set

$$\Gamma = \{\gamma_1, \dots, \gamma_n\}$$

of simple closed non-peripheral curves in  $\mathbb{C} \setminus P_f$  is called a Levy cycle if for any  $\gamma_i \in \Gamma$ , there is a simple closed non-peripheral curve component  $\gamma'$  of  $f^{-1}(\gamma_i)$  such that  $\gamma'$  is homotopic to  $\gamma_{i-1}$  (we identify  $\gamma_0$  with  $\gamma_n$ ) rel  $P_f$  and  $f : \gamma' \rightarrow \gamma_i$  is a homeomorphism. Following a result in [HSS], we then have that

**Corollary 4** ([HSS]). *Suppose  $f$  is a  $(0, 1)$ -topological exponential map with finite post-singular set. Then  $f$  has no Levy cycle if and only if  $f$  has bounded geometry.*

We believe a similar result holds for all post-singularly finite maps in  $\mathcal{T}E_{p,q}$  but we do not have a proof at this time.

## REFERENCES

- [Al] L. Ahlfors, Complex Analysis, Third Edition, McGraw-Hill, 1979.
- [CJ1] T. Chen and Y. Jiang, Bounded geometry and characterization of rational maps. In preparation.
- [CJ2] T. Chen and Y. Jiang, Canonical Thurston obstructions for sub-hyperbolic semi-rational branched coverings. arXiv:1101.2285v2 [math.DS]

- [CJK] T. Chen, Y. Jiang and L. Keen, Bounded geometry and families of meromorphic functions with two asymptotic values. arXiv:1112.2557v1 [math.DS]
- [DH] A. Douady and J. H. Hubbard, A proof of Thurston's topological characterization of rational functions. *Acta Math.*, Vol. **171**, 1993, 263-297. MR1251582 (94j:58143)
- [HSS] J. Hubbard, D. Schleicher, and M. Shishikura, Exponential Thurston maps and limits of quadratic differentials. *Journal of the American Mathematical Society*, Vol. **22** (2009), 77-117. MR2449055 (2010c:37100)
- [Ji] Y. Jiang, A framework towards understanding the characterization of holomorphic maps. Appeared in the same volume.
- [N] R. Nevanlinna, *Analytic Functions*, Springer 1970. MR0279280 (43 #5003)
- [Pi] K. M. Pilgrim, Canonical Thurston obstruction. *Advances in Mathematics*, **158** (2001), 154-168.
- [T] W. Thurston, The combinatorics of iterated rational maps. Preprint, Princeton University, Princeton, N.J. 1983.
- [Z] S. Zakeri, On Siegel disks of a class of entire maps. *Duke Math. J.*, Vol. **152** (2010), no. 3, 481-532. MR2654221 (2011e:37102)
- [ZJ] G. Zhang and Y. Jiang, Combinatorial characterization of sub-hyperbolic rational maps. *Advances in Mathematics*, **221** (2009), 1990-2018. 1990-018. MR2522834 (2010h:37095)

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